ON THE AUTOMATED OPTIMAL DESIGN
OF
CONSTRAINED STRUCTURES

BY

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DEDICATION

To my grandmother, Mrs. Florence Hornbuckle, and my wife, Carolyn. Without the love, confidence, and patient understanding of Granny Hornbuckle and Carolyn my graduate studies would never have been attempted.

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Pontryagin's Maximum Principle is applied to the optimal design of elastic structures, subject to both hard inequality constraints and subsidiary conditions. By analyzing the maximum principle as a non-linear programming problem, an explicit formulation is derived for the Lagrangian multiplier functions that adjoin the constraints to the cost functional. With this result the usual necessary conditions for optimality can themselves be used directly in an algorithm for obtaining a solution.

A survey of general methods and problems in the optimal design of elastic structures shows that there are two general problem types depending upon whether or not the cost functional is an eigenvalue. An example problem of each type is included with the solutions obtained by the method of quasilinearization. In the first example, a minimum deflection beam problem, classical Maximum Principle techniques are used. The eigenvalue problem is exemplified by the maximization of the buckling load of a column and uses the explicit multiplier function

formulation mentioned above. Since the problem considered is conservative, it is therefore described mathematically by a self-adjoint system; under this condition it is shown that the minimum weight problem is identical to the maximum buckling load problem.

In order to demonstrate the theory for the programming techniques used, the beam problem is also solved by using a finite element representation of the structure. From a comparison to the maximum principle solution it is shown that the form of the optimal solution obtained is dependent upon the magnitude of the tolerance used with the numerical solution scheme. Furthermore, it is shown that convergence by the quasilinearization algorithm is related to the respective curvatures of the initial guess and the solution.

Recommendations for additional investigations pertinent to this study are also included.

CHAPTER I

INTRODUCTION

1.0 Survey Papers

It is exceedingly difficult to write a general introduction to the field of structural optimization for two basic reasons: (i) there is no conventionally accepted nomenclature, and (ii) there is also no conventionally accepted classification of problem types or characteristics. In marked contrast, when one considers the calculus of variations, "cost functional, system equations, kinematic and natural boundary conditions, adjoint variables, Hamiltonian, etc.," all have well-defined, universally accepted meanings. Additionally, there is no confusion when speaking of the problem types of Mayer, Lagrange, and Bolza. This common language and categorization of problems does not exist in structural optimization.

Instead, the field tends to branch and fragment into very specialized sub-disciplines that are oriented towards applications. While these branches are related to the general field, the techniques and methods of one branch can seldom be applied to another. Moreover, as a result of the tendency to an applications orientation, solutions are generally ad hoc and not useful for other problems even within the same branch. The lack of any definitive unification of the subject cannot be blamed either on being recently developed or in receiving too little attention.

This is readily seen by considering the survey papers described in the following paragraphs.

The earliest comprehensive survey paper is Wasiutynski and Brandt (1963). Although their excellent historical development is dominated by Russian and Eastern European references, the authors do include a higher percentage of papers by Western authors than is encountered in the typical paper from Eastern Europe. A more fundamental criticism is that too little is said regarding problem types or solution methods.

Chronologically, the next survey paper is Gerard (1966). The theme of this review is aerospace applications, with a particular orientation to the design-management, decision-making process. Most of the papers cited treat specialized aerospace structures and applications; however, the author does try to generalize by introducing a design index D, a material efficiency parameter M, and a structural efficiency parameter S. After defining the expressions for M and S corresponding to several structural elements, design charts are presented which show regions of possible application for various materials. Unfortunately the design charts do not satisfy expectations aroused by the introduction of the three general parameters.

Rozvany (1966) presents a similar paper pertaining to structures in civil engineering. This paper is less comprehensive and more oriented towards specific structural applications than either of the preceding survey papers. The author postulates several "interrelated quantities (or parameters)" which could perhaps be used to generalize

structural optimization into a more rational methodology. However, these quantities--loading (L), material (M), geometry (G), initial behavior (IB), and design behavior (SB)--are only applied to an abstract discussion of concepts.

Barnett (1966) has a readable short survey of the field that dwells more upon theoretical aspects. He postulates a general problem in which the cost is minimized subject to a system in equilibrium with its loads, while "behavior constraints on strength, stiffness, and stability" are satisfied. Uniform strength design introduces the discussion of optimal trusses; virtual work theorems that were derived originally for trusses are then applied to simultaneous plastic collapse problems. Following a brief discussion of the plastic collapse load bounding theorems, there is a short treatment of elastic stability problems and material merit indices. Barnett's stiffness design example exhibits several important features worth noting. Specifically, the example is to minimize the weight of a beam subject to some given load where the deflection at a certain point is specified. Virtual work is used to handle the subsidiary deflection condition. Necessary conditions are obtained from the calculus of variations, but more significantly, the Schwarz inequality provides a sufficient condition for global optimality. Barnett concludes his survey with a section stating that multiload designs satisfying "all three behavior criteria" are more easily solved in "design space" by mathematical programming techniques described by Schmidt (1966).

As a sequel to the comprehensive survey (1638 to 1962) by Wasiutynski and Brandt, Sheu and Prager (1968b) present a complete review

of developments from 1962 to 1968. This paper contains three major sections: general background, methodology, and specific problems. In the first section they state that the "well-posed problem of optimal structural design" requires specification of the

(1) purpose of the structure (load and environment),

(2) geometric design constraints (limits to design parameters),

(3) behavioral design constraints (limits to the "state" of the structure),

(4) design objective (cost functional).

The methodology section is not noteworthy, but the third section lists what is in their opinion the specific problem types: static compliance, dynamic compliance, buckling load, plastic collapse load, multipurpose/multiconstraint structures, optimal layout (e.g., trusses), reinforced/prestressed structures, and from the background section, probabilistic problems. Their concluding remarks succinctly summarize the paramount difficulty of the subject: realistic problems are too complicated for precise analytical treatment. While progress is being made in analytical treatment of simple structures, the authors opine that realistic problems require mathematical programming techniques. However, they do feel that analytical treatment is desirable to provide "a deeper insight into the analytical nature of optimality."

A related survey by Wang (1968) on distributed parameter systems, ". . . whose dynamical behaviors are describable by partial differential equations, integral equations, or functional differential equations," consists entirely of a bibliography. While not pertinent to the dissertation, it is mentioned here for completeness.

Prager (1970) provides another survey which is not comprehensive, nor does he present any new results as claimed. For example, Prager and Taylor (1968) used the principle of minimum potential energy and the assumption that stiffness is proportional to specific mass to prove global optimality. It could hardly be called a new development in 1970. At the same time, the author does present an excellent example of a multipurpose optimal design problem. Prager also treats "segment-wise constant" approximations and the optimal layout of trusses.

The final survey paper, Troitskii (1971), is an unusual review of methods in the calculus of variations. Whereas some of the earlier surveys present lengthy lists of references but contain little methodology or theory, this survey is just the opposite. Part of what makes it unique is that the author believed only eight articles merited citation--all of them by Troitskii. This shortcoming is more than overcome by a thorough classification of optimal control problems in the calculus of variations. Troitskii bases the classification on "certain characteristics of control problems": types of constraints, properties of the governing dynamical equations of the system, type of cost functional, and possible state discontinuities. From these four criteria he postulates five principal classes of problems; however, it is the criteria that are important and not the specific problem type.

By comparing what the authors of the aforementioned surveys believe to be the important types of problems, it is readily apparent that there is little agreement on which characteristics of structural optimization problems are significant.

1.1 Historical Development: Optimal Columns

The beginning of structural optimization is generally attributed to Galileo's studies in 1638 of the bending strength of beams. According to Barnett (1968), Galileo considered a constant-width cantilever beam under a tip load as part of a study of "solids of equal resistance." In requiring the maximum stress in each cross section to be constant throughout the beam, the height must be a parabolic function of position along the beam. While this appears to be the origin of the field, a problem that received more attention is the buckling of a column.

Using the newly developed calculus of variations, in 1773 Lagrange attempted to apply variational techniques to the problem of finding that distribution of a homogeneous material along the length of a column which maximizes the buckling load. Truesdell (1968) relates that through an insufficient mathematical formulation Lagrange showed the optimal form to be a circular cylinder. Clausen (1851) provides the earliest known solution to this problem for the simply supported case. As described in Todhunter and Pearson (1893, pp. 325-329), Clausen minimized the volume of the column with the differential equation for buckled deflection treated as a subsidiary condition. Assuming all cross sections to be similar, after several variable transformations and complicated manipulation, he obtained an implicit, analytical solution.

The next development was Greenhill (1881), according to Keller and Niordson (1966). Greenhill determined the height of a uniform prismatic column, beyond which the column buckled due solely to its own weight. Timoshenko and Gere (1961) reproduce the solution in which the

deflection is expressed as the integral of a Bessel function of the first kind (of the negative one-third order).

Blasius (1913) introduces his paper with a uniform strength and a minimum deflection beam problem. For a given load and amount of material, the cross-sectional area distribution is determined which maximizes the buckling load of a circular column. The solution is identical to that obtained by Clausen. In addition, Blasius also obtained solutions for columns having rectangular cross sections and discussed the effect of different boundary conditions on the results.

For the next few decades, structural optimization appears to have been directed towards applications in the aircraft industry, where aircraft structural problems and results are presented in the format of a design handbook. Feigen (1952) is a good example of this, considering the buckling of a thin-wall column. Given a constant load and wall thickness, he required the variable inside diameter to be chosen such that the buckling load is maximized. Wall thickness is selected to make local buckling and Euler buckling occur at the same load. Solid tapered columns having blunt ends are also treated for assumed stiffness distributions.

Renewed interest was aroused by Keller (1960), who examined the problem from the point of view of the theory of elasticity, and in choosing the cross-sectional shape to give the maximum stiffness. Neglecting the weight of the column, he obtained via the former that twisting the column does not affect the buckling. Of all convex cross sections, the equilateral triangle is shown to have the largest second

moment of area relative to a centroidal axis. Hence, from the definition of buckling load, the "best" cross-sectional shape is the equilateral triangle. Keller also obtained Clausen's implicit, analytical solution. Subsequently, Tadjbakhsh and Keller (1962) generalized the problem to a general eigenvalue problem and boundary conditions subject to a subsidiary equality constraint. The latter corresponds to specifying the volume (or weight) of material to be distributed in an optimal manner. Using the Hölder inequality they demonstrate global optimality of the eigenvalue for the hinged-hinged column.

Keller and Niordson (1966) examine the case of a vertical column fixed at the base, subject to a vertical load at the tip and the column's own weight. It is also assumed that all cross sections are similar. Their approach is to state the problem as a simultaneous, dual optimization of the Rayleigh quotient. The eigenvalue is minimized with respect to the eigenfunction and maximized with respect to cross-sectional area distribution. Specifying the volume of material available is treated as a subsidiary equality constraint. From the maximum lowest eigenvalue the maximum height at buckling is determined. Solutions are obtained by an iterative technique employed with integral equations.

In a brief note, Taylor (1967), suggests that energy methods may link optimum column problems to the traditional eigenvalue problems of mechanics. Prager and Taylor (1968) classify problems in optimal structural design and demonstrate global optimality using energy principles. Unfortunately their assumption of thin-wall construction limits the results to structures where the stiffness is proportional to the specific

mass density. The consequence of this assumption is that in the energy formulation the resulting control law and governing equations are decoupled, and hence easily solved. Huang and Sheu (1968) apply this same thin-wall assumption to the problem treated by Keller and Niordson. However, the former seek the maximum end load instead of the maximum height. The authors also attempt to limit the maximum allowable stress and obtain solutions by a finite-difference method. Further discussion of sandwich (thin-wall construction) columns is given by Taylor and Liu (1968). Basically, this paper is an elaboration of the techniques described by Prager and Taylor when applied to columns. Extensive results are provided for various cases.

Post-buckling behavior for columns subject to conservative loads is considered by Gajewski and Zyczkowski (1970). A nonconservative problem is treated by Plaut (1971b). The first of these two papers is lengthy but is much too narrow in scope to be particularly useful. In the second paper, the Ritz method is applied to an energy functional, obtaining the "best" form of the assumed approximation to the optimal solution.

1.2 Historical Development: Optimal Static Beams

That beam problems played a role in the early developments of structural optimization has already been indicated in the preceding section. No attempt is made in what follows to present a complete history, but merely to outline the type of problems that have been considered.

Opatowski (1944) has an outstanding paper that deals with cantilever beams of uniform strength. Besides providing numerous references to earlier studies, the author treats the problem with impressive mathematical rigor. The beam is considered to deflect under its own weight and a transverse tip load; bending deflection is described by a Volterra integral equation which is solved exactly for various assumed classes of variable cross-sectional geometry. This paper is representative of earlier papers in that it contains extensive analysis and analytical results, but little numerical data. Barnett's work (1961) and its sequal (1963a) apply the calculus of variations to more realistic I-beams. One problem considered is maximizing the weight subject to general, unspecified loads for a specified deflection at a given point. The Schwarz inequality is used to derive a sufficient condition for global optimality. Also included is a comparison of uniform strength beams to the minimum deflection beam for several different cases of applied load and geometry. The paper is concluded with various minimum weight design examples in which both bending and shear stiffness are considered.

Haug and Kirmser's (1967) work is one of the most comprehensive studies of minimum weight beam problem published. While it may succeed in handling any conceivable problem and in employing the most realistic stress constraints, this very generality requires so many variables and conditions that the mathematics is complicated almost beyond reason.

Another study of minimum weight beams (Huang and Tang, 1969) is important for several reasons. By dividing the beam into many segments

having constant properties, and determining the necessary conditions that must be satisfied by every segment, it appears that the authors are using the same methods that were used to solve the Brachistochrone problem in the seventeenth century.* In their treatment, multiple loads must produce a specified deflection at a given point; these "segmentwise constant" approximations and multiple load problems have recently received more attention. Of further interest in this paper is the derivation of the multiple load optimality condition using Pontryagin's Maximum Principle.

While no new results or techniques are contained in Citron's (1969, pp. 154-166), the author gives a very readable minimum weight beam example. The problem is simple, described in detail completely, and provides an excellent example of how control theory is applied to an optimal problem in structural design. An analysis of intermediate beams which may form plastic hinges is provided by Gjelsvik (1971). It shows that if hinges are placed at all points of the beam where the bending moment is zero, this makes the plastic or elastic minimum weight beam statically determinate. Both the elastic and plastic beam designs are shown to be fully stressed, i.e., are uniform strength beams.

Application of generalized vector space techniques is characteristic of more recent papers. Bhargava and Duffin's (1973) is such a study. It treats the maximum strength of a cantilever beam on an elastic

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*Since these early papers are not readily available, this statement is based upon descriptions of them given by historical references cited in Appendix A.

foundation subject to an upper bound on weight. Although involving more advanced mathematics than normally required by variational techniques, vector space methods may also provide more powerful analytical tools.

1.3 Historical Development: Optimal Dynamical Beams

The title of this section is a misnomer. On the basis of the related literature, a more realistic terminology would be "Optimal Quasistatic Beams." Papers dealing with the optimization of dynamical beams ultimately contain some assumption or given condition that effectively transforms the problem to an equivalent static case. Simple harmonic motion is frequently assumed to remove time dependence from the governing equation. For example, Barnett (1963b) minimizes the tip of deflection of a cantilever beam accelerating uniformly upwards, with the total weight of the beam specified. The optimal solution for cross-sectional area distribution is specified by a nonlinear integral equation solved by successive approximations. Dynamics enters the problem only as a time invariant inertia load which converts the problem to a static beam subject to a body force.

Niordson (1964) finds the tapering of a simply supported beam of given volume and length which maximizes the fundamental frequency of free vibration. Assuming that all cross sections are similar, Niordson expresses the desired frequency as the Rayleigh quotient. This is obtained by the equation for spatial dependence associated with the usual separation of variables technique. In this case it is assumed

that deflection, shear, and rotational inertia effects are small. Solution of the conditions for optimality were obtained by successive approximations. This approach results in a problem identical in form to the eigenvalue problem associated with maximizing the Euler buckling load of a column.

A very specialized problem is treated in Brach (1968). Cross-sectional area and stiffness are proportional, and have both upper and lower bounds; material properties and length of the beam are specified constants. The object is to make the fundamental frequency of free vibration stationary, without any constraints on the weight. Instead of using the Rayleigh quotient, Brach uses the total potential energy. His solution method is ad hoc and not generally useful. A more useful approach is described by Icerman (1969) for structures subject to periodic loads. Necessary and sufficient conditions for minimum weight are obtained from the principle of minimum potential energy. Amplitude and frequency of the applied load is specified as well as "dynamic response," defined as the potential energy associated with the load amplitude displaced a distance equal to the displacement amplitude at the point of application. It is also required that the load's frequency be less than the fundamental natural frequency. Subject to these constraints, the structure's weight is to be minimized. Trusses and segmentwise constant approximations are also treated. Once again a dynamical problem is effectively transformed into a quasistatic problem.

Realistic treatment of the dynamic response of beams subject to time dependent loads is given in Plaut (1970). An upper bound on the

response of the beam is obtained from the "largest possible displacement of the beam under a static concentrated unit load." This inequality is determined from the time derivative of the total energy of the beam and the Schwarz inequality. Plaut minimizes the upper bound on the response for specified total weight and relationship between specific mass and stiffness. Despite the consideration of a truly dynamical problem, this approach has two weaknesses indicated by the author. First, the upper bound need not be close to the exact answer; secondly, there is no demonstration that minimizing the upper bound also minimizes the actual response of the beam subject to dynamic loading.

Another paper worthy of mention is Brach and Walters (1970). They maximize the fundamental natural frequency which is expressed as a Rayleigh quotient that includes the effect of shear. Standard variational methods are employed to derive necessary conditions, but no solutions nor examples are given. The authors do, however, suggest using the method of quasilinearization. This paper is another example of a quasistatic, dynamical beam problem.

1.4 Scope of the Dissertation

This dissertation is primarily concerned with the application of Pontryagin's Maximum Principle to problems in structural optimization. Only elastic materials are considered; however, various types of constraints are treated.

The theoretical development of the dissertation pertains to problems described by an ordinary differential equation but is based upon a numerical technique normally used with systems described by

a finite number of discrete quantities. For this reason, examples are included for both types of systems.

No new solution techniques have been developed for the nonlinear two point boundary value problems which characteristically arise in optimization problems for continuous systems. Solutions are obtained by a standard quasilinearization method. However, a modified "feasible direction" numerical algorithm for use with discrete systems is described and an example included to demonstrate its operation. This serves to illustrate the application of the theory on which the algorithm is based to the theoretical development associated with continuous systems. Furthermore, it provides a comparison between the solution to a problem described as a continuous system, and alternately by a discrete approximation.

Additionally, it is the intent of this dissertation to replace some of the confusion in classification of problem types contained in the various survey papers with an organization based upon mathematical attributes. The result is a logical approach to the formulation of optimization problems for elastic structures.

CHAPTER II

GENERAL PROBLEMS AND METHODS IN STRUCTURAL OPTIMIZATION

2.0 Introduction

The first chapter presents a broad view of structural optimization and the historical development of two general types of problems that are used as examples in later chapters. Before the mathematical theory is developed in the next chapters, general problems and methods in structural optimization itself are briefly outlined in this chapter. The classification of problem types vis-à-vis mathematical attributes is discussed first. This is followed by short descriptions of the major methods of structural optimization for both continuous and discrete systems. Appropriate references are cited for each section.

2.1 Problem Classification Criteria

Perhaps the major source of difficulty in classifying structural optimization problems lies in the translation from the physics involved to a mathematical representation. A single physical concept when transformed to mathematics may become more than one mathematical attribute. For example, consider the class of conservative problems. Since energy is conserved this immediately prohibits dissipative materials, time varient constraints, and nonholonomic constants. More important, consider the following statements from Lanczos (1962, p. 226):

> . . . all the equations of mathematical physics which do not involve any energy losses are deducible from a "principle of least action," that is the principle of making a certain scalar quantity a minimum or a maximum . . . all the differential equations which are self-adjoint, are deducible from a minimum-maximum principle . . . and vice versa.

However, it is shown in Chapter V that to be self-adjoint systems places requirements on both the differential equation and the boundary condition. Thus, the single physics attribute of being a conservative system is described mathematically by expressions involving the cost functional, the governing system of differential equations, boundary conditions, and constraints.

As a result of this lack of similarity in descriptions, a choice must be made as to which realm will be used for classification of problems. Since all problems are ultimately transformed to mathematics, mathematical characteristics are selected as the criteria. On the basis of survey paper contents and the many related papers, it is felt that the proper (not necessarily the best, nor all inclusive) characteristics for classification of problem types are:

(i) cost functional

(ii) system equation and boundary conditions

(iii) control constraints

(iv) behavioral constraints (state and/or control constraints)

Subsequent discussion is in terms of these four characteristics. Exceptions to these descriptors are readily acknowledged, e.g., whether the problem is deterministic or probabilistic. An example of the latter may be seen in Moses and Kinser (1967). These exceptions do not serve

as negating counterexamples but instead indicate the requirement for additional descriptors and verify the difficulty of the task, suggesting the need for further comprehensive study.

2.1.0 Problem Classification Guidelines

In the following sections each criterion is briefly discussed. Some of the various characteristics of each are mentioned, and where appropriate references exist the citation is given. It must again be emphasized that the following is not all inclusive; it is an attempt to categorize the types of problems existing in the literature according to the four mathematical descriptors postulated. Moreover, the descriptors are not discussed in the order given but instead are treated in the order normally encountered during a problem solution.

2.1.1 Governing Equations of the System

The immediate question to be answered is whether the structural system is described by a set of continuous functions or a set of discrete constants. Bending deflection of simple structural elements is an example of the former; design of trusses is a good example of the latter. In general, variational techniques are employed with continuous systems while mathematical programming techniques are most frequently applied to the discrete systems. However, variational methods can be applied to the approximation of continuous systems by discrete elements. This usually takes the form of either a finite element or "segmentwise constant" approximation of the continuous structure. References treating the different systems described above are included in the section on methods.

2.1.2 Constraints

Two of the descriptors are postulated to be control constraints and behavioral constraints. A further consideration is whether the constraint is defined by an equality relationship or by an inequality expression. Equality constraints are handled by a long-known technique entitled "isoperimetric constraints." Valentine's (1937) work is known to contain the initial development of a technique for converting inequality constraints to equality constraints. The introduction of slack variables increases the number of variables in the problem to be solved but at the same time permits all of the isoperimetric techniques to be used. A detailed application of this approach is presented in Appendix B. It should also be noted that isoperimetric constraints are sometimes referred to as "accessory or subsidiary conditions."

Most real structural optimization problems possess an isoperimetric constraint as well as inequality constraints dependent upon the control $\underline{u}$ and/or the state $\underline{x}$ of the structural system.* Typically the control constraints are the result of geometrical limitations or restrictions to the types of available materials. Behavioral constraints are related to the state of the system and may depend solely upon the state (of deformation), or in the case of most stress constraints, jointly upon the state and control variables. With this distinction the constraints may be classified vis-à-vis the two criteria and optimal control characteristics as follows:

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*By convention all vectors are column vectors unless indicated otherwise.

(i) unconstrained

(ii) $\phi(\underline{u}) \leq 0$ control constraint

$$\left.\begin{array}{ll}\text{(iii)} & \phi(x,u) \leq 0 \\ \text{(iv)} & \phi(\underline{x}) \leq 0\end{array}\right\} \text{ behavioral constraints}$$

All of this discussion pertains to both continuous and discrete systems.

No references for unconstrained optimization problems are given. It may be that in some problems the unconstrained structural optimization solutions have either infinite stiffness and finite weight, or finite stiffness and zero weight. A discussion of this can be found in Salinas (1968, pp. 23-26).

Investigation of control constraints led to the development of the maximum principle. Although reserving discussion of the method for a later section, the classical reference detailing the derivation of the principle is given here for completeness. Rozonoér (1959) treats control constraints but only as related to the development of the maximum principle.

Most of the literature concerns either bounded control problems or a more general form of constraint which can be classified as a behavioral constraint. The latter is a mixed constraint which depends upon both the state and control variables. Breakwell (1959) is a lucid paper dealing with this type of constraint. References which treat state and/or mixed constraints are Bryson et al. (1963), and Speyer and Bryson (1968). Constraints which depend upon only the state are not treated in this dissertation; an example of such a constraint is to determine the optimal solution for some problem subject to an upper bound on deflection of the structure at any point.

2.1.3 Cost Functionals

There are two basic types of cost functionals that occur in the field of structural optimization. They were first identified by Prager (1969) but not for the proper reasons. Using Prager's notation, they are

$$J = \mathrm{Min} \int_V F(\psi)\, dV$$

$$J_Q = \mathrm{Min} \frac{\int_V G(\psi)\, dV}{\int_V H(\psi)\, dV}$$

where F, G, and H are scalar functionals of ψ. The latter functional represents a Rayleigh quotient associated with an eigenvalue problem. It can be reduced to the first type of functional shown above by choosing a normalization of the eigenfunction such that the numerator equals unity for all admissible variations (see section 5.3). This normalization is thereafter treated as a subsidiary constraint.

What actually distinguishes the second functional from the first is not that the functional is a quotient, but that the extremization of an eigenvalue requires a dual extremization (see section 5.3). In terms of a state $\underline{x}$ and control $\underline{u}$, the fundamental eigenvalue is given by minimization of the Rayleigh quotient with respect to the eigenfunction $\underline{x}$, or

$$J_Q = \mathrm{Min}_{\underline{x}} \frac{\int_V G(\underline{x},\underline{u})\, dV}{\int_V H(\underline{x},\underline{u})\, dV}$$

where $\underline{u}$ represents some specified design parameter. If the desired result is to maximize the cost J_Q with respect to all admissible $\underline{u}$, it is observed that a second extremization is required; for example, see

Keller and Niordson (1966). Thus, a more appropriate manner of classifying cost functionals is on the basis of whether the problem statement implies a single or a double extremization. Hence the two basic types of cost functionals encountered in structural optimization are

$$J = \underset{\underline{u}}{\mathrm{Min}} \int_V F(\underline{x},\underline{u})\, dV$$

$$J_Q = \underset{\underline{u}}{\mathrm{Max}}\ \underset{\underline{x}}{\mathrm{Min}} \frac{\int_V G(\underline{x},\underline{u})\, dV}{\int_V H(\underline{x},\underline{u})\, dV}$$

where $\underline{x}$ must satisfy an equilibrium condition of the state, and $\underline{u}$ is subject to some admissibility requirements.

There is a special case related to these two in which the weight is to be minimized for a specified eigenvalue. This problem is treated in Icerman (1969) with a mathematical discussion of such a variational problem presented in Irving and Mullineux (1959, p. 394). In terms of the two cost functionals, the special case is

$$J = \underset{\underline{u}}{\mathrm{Min}} \int_V \{G(\underline{x},\underline{u}) - J_Q H(\underline{x},\underline{u})\}\, dV$$

where J_Q is a specified constant. This approach is frequently employed in eigenvalue problems to avoid the inherent difficulties associated with the dual extremization problem.

There have also been many papers published that consider "multi-purpose structures," e.g., Prager (1969), Martin (1970), and Prager and Shield (1968). The cost function for such problems is defined as

$$J = \sum_{i=1}^{k} a_i J_i(\underline{x},\underline{u})$$

where a_i are positive constants, serving as weighting parameters. While perhaps demonstrating much potential, no significant results obtained with this approach have so far been published. What problems have been solved are too simple; indeed the authors indicate the need for using a discrete approximation and mathematical programming techniques in realistic applications.

A subject closely related to "multi-purpose structures" is that of multiple constraints. It is mentioned here only because most papers on the latter also include the former--see Martin (1971). The idea of multiple constraints is not new; in both variational and mathematical programming fields there exist standard techniques for handling multiple constraints.

A recent Russian paper (Salukvadze, 1971) suggests an alternate to the "multi-purpose" cost functional. Instead of treating a vector functional that requires the choosing of weighting coefficients, it is suggested that the several functionals be combined into one. Given a system and vector cost functionals

$$\underline{\dot{x}} = \underline{f}(\underline{x},\underline{u},t)$$

$$J_i[\underline{u}] = J_i(\underline{x},\underline{u},t) \qquad i = 1,\ldots,k$$

Let $\underline{u}_{OPT}^{(i)}$ denote the optimal solution for which J_i assumes the optimal value on the trajectory of the system. For each of the J_i there is a different $\underline{u}_{OPT}^{(i)}$. These k values J_i can be thought of as components of

a vector $\underline{r}^*$ where[†]

$$\underline{r}^* = \left\{J_1\left[\underline{u}^{(1)}_{OPT}\right] \ \cdots \ J_k\left[\underline{u}^{(k)}_{OPT}\right]\right\}^T$$

For any arbitrary $\underline{u}$ the result

$$\underline{r}[\underline{u}] = \{J_1[\underline{u}] \ \cdots \ J_k[\underline{u}]\}^T$$

is just some vector functional.

Vector $\underline{r}^*$ represents a constant point in the space of $(J_1,\ldots,J_k)$. Since no choice of $\underline{u}$ can optimize all of the J_i simultaneously, that is, to attain the point $\underline{r}^*$ in J_i-space, the best alternative is to minimize the distance between $\underline{r}[\underline{u}]$ and $\underline{r}^*$. That distance is defined by the Euclidean norm. To avoid the question of inconsistent dimensions the functionals are reduced to dimensionless form. Thus,

$$J[\underline{u}] = \sum_{i=1}^{k}\left[\frac{J_i[\underline{u}] - J_i\left[\underline{u}^{(i)}_{OPT}\right]}{J_i\left[\underline{u}^{(i)}_{OPT}\right]}\right]^2$$

and $\underline{u}_{OPT}$ is that function $\underline{u}$ which minimizes the functional $J[\underline{u}]$. Mathematically speaking,

$$\underline{u}_{OPT} = \text{ARGMIN}\ \{J[\underline{u}]\}$$

This type of vector cost functional is much more appealing than the type treated in the papers on multi-purpose structures. It also suggests an entirely new field of study: the more realistic

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[†]Superscript "T", e.g., $\underline{u}^T$, denotes the transpose of the vector $\underline{u}$.

choice of cost functionals. The mathematics of a problem seldom accommodates financial considerations. For example, the design which requires the least material may reduce the cost of materials at an overwhelming expense in manufacturing or fabrication. When aesthetic appeal and environmental impact are included--as must be done in any real, commercial application--the selection of an appropriate cost functional is an almost insurmountable task. However, a simple extension of Salukvadze's composite cost functional may reduce the difficulties to operations research considerations.

In problems where it is desired to optimize simultaneously several different functionals, not all having the same dimensions, the concept of a generalized inner product may prove useful. It is defined in terms of a metric operator A; for some general vector $\underline{z}$

$$\|\underline{z}\|^2 \triangleq (\underline{z},\underline{z})_A = \underline{z}^T A \underline{z}$$

and symbol "$\triangleq$" means "is defined by." With reference to the vector cost functional, A represents a set of scale factors which converts all of the separate cost functionals to a common dimension. This is where the operations research enters--relating material expense to fabrication to sociological considerations and so forth--to determine the metric A. For a vector $\underline{\varepsilon}$ whose elements are functionals,

$$\underline{\varepsilon} = \underline{r}[\underline{u}] - \underline{r}^*$$

where $\underline{r}[\underline{u}]$ and $\underline{r}^*$ are defined above, the composite cost functional is

$$J[\underline{u}] = (\underline{\varepsilon}, A\underline{\varepsilon})$$

The weakness in this method is that an optimal solution must be obtained for each individual cost functional prior to attempting a solution to the composite problem. Additionally, some of the more abstract cost objectives may be difficult to quantify in a meaningful manner. Despite these shortcomings this approach does suggest interesting applications.

2.2 Methods: Continuous Systems

The problems characterized by mathematical functions, in contrast to those represented by a set of discrete constants, are normally treated by variational techniques. Many books on this subject have been published; the better authors include Elsgolc (1961), Gelfand and Fomin (1963), Dreyfus (1965), Hestenes (1966), Denn (1969), Luenberger (1969), and Bryson and Ho (1969). An excellent summary paper is available in Berg (1962).

To see how these techniques are applied, three papers are recommended. The first is Blasius (1913), which provides sufficient detail and explanation to make it quite worthwhile. Although it does include several examples that involve subsidiary conditions, no inequality constraints are treated. An example that includes inequality constraints to the control variable is coontained in Brach (1968). A much more general application of variational principles is presented in Oden and Reddy (1974). In this paper a dual-complementary variational principle is developed for a particular class of problems. It is shown that the canonical equations obtained are the Euler-Lagrange equations for a certain functional.

2.2.0 Special Variational Methods

Besides the ordinary variational method, more specialized techniques have been developed to the point where they are recognized as independent methods in their own right. In the following sections these methods are identified and a number of representative references given.

2.2.1 Energy Methods

The oldest of these methods is the energy method. It originated with the principle of minimum potential energy, and was later extended to include the representation of eigenvalues through the energy functional. A good discussion of the former is available in Fung (1965) or Przemieniecki (1968); the best general treatment of the latter is available in either Gould (1957, Chapter 4) or in Mikhlin and Smolitskiy (1967, Chapter 3).

The principle of minimum potential energy is frequently used with simple problems to prove that a necessary condition for optimality is also sufficient. Prager and Taylor (1968) contains such a proof for the global, maximum stiffness design of an elastic structure of sandwich construction; two papers that also consider this problem are Huang (1968) and Taylor (1969). Specific application of the energy method to an eigenvalue problem is demonstrated in Taylor and Liu (1968). A much more general discussion of the energy method is provided in Salinas (1968). Further extensions of the method are presented in Masur (1970), in which the principle of minimum complementary energy is applied to problems of the optimum stiffness and strength of elastic

structures. In these problems a necessary condition for optimality is that the strain energy density be constant throughout the structure. This condition is also sufficient for optimality in certain classes of structures that satisfy a specific relationship between the strain energy density and design variables.

Many of the energy problems belong to the class of problems having quadratic cost functionals. The significance of this characteristic is that the Euler-Lagrange equations derived from such functionals are linear.

A more recent energy method development is the concept of "mutual potential energy." Mutual potential energy techniques resemble those of the principle of minimum potential energy. In both methods a cost functional is defined over the entire domain occupied by the structure and optimized with respect to the control variable. If it is desired that the optimal solution be required to have a specified deflection at a certain point, this condition corresponds to a subsidiary state constraint when using the principle of minimum potential energy. The mutual potential energy method incorporates this type of localized constraint into the cost functional which is defined over the entire domain of the structure. By itself this alone is advantageous; however, for certain types of problems the mutual potential energy method also provides both a necessary and sufficient condition for global optimality. In the way of a critical comment, either the method has received too little attention, or else it does not efficiently handle problems more difficult than the simple examples presented.

Four papers that are representative of the literature associated with this method are Shield and Prager (1970), Chern (1971a, 1971b), and Plaut (1971c).

Another recent development is that the class of problems for which energy methods is applicable has been expanded to include certain types of nonconservative systems. Together with the mutual potential energy concepts, this suggests that perhaps the classical energy method is a special case of a more generalized method. If a technique can be developed which uses the adjoint variables to transform a general nonconservative system into an equivalent self-adjoint form, the method might be deduced. Some papers pertaining to the subject are Prasad and Herrmann (1969), Wu (1973), and Barston (1974).

2.2.2 Pontryagin's Maximum Principle

There are many textbooks which derive, explain, and give examples for the maximum principle. The original (Pontryagin et al., 1962) requires a knowledge of functional analysis. A condensed form of this same material is available in Rozonoér (1959). Denn (1969) provides another point of view in which the principle is derived from Green's functions. In this manner, the sensitivity to variations is readily observed. To understand Denn's treatment requires only a knowledge of the solution of differential equations.

Shortly after Pontryagin's book was published, many papers devoted to the theoretical aspects of the maximum principle were published. Some of the more readable ones are Kopp (1962, 1963), Roxin (1963), and Halkin (1963). Another early paper (Breakwell, 1959)

appears to be a completely independent derivation of the maximum principle. Although quite general in the mathematical sense, the examples presented are trajectory optimization problems and not a general type of mathematical problem. This may be an explanation for what seems to be a lack of recognition for a significant achievement.

The application of PMP to problems in structural optimization is relatively recent. When the method is used, one of two difficulties is often encountered. The first is errors in the formulation of the optimal control problem; the second is that once a well-posed, nonlinear two-point boundary value problem (TPBVP) is obtained, it is difficult to solve. An example of the first is provided by Dixon (1968)--the correction was given in Boykin and Sierakowski (1972). De Silva (1972) provides a clear presentation on the application of PMP to a specific problem, but includes no data because a solution could not be obtained. Despite the failure to determine the solution, this paper is worthwhile for its lucid discussion of the PMP application. Another paper that gives a good specific application of PMP is Maday (1973). Although much analysis is presented very little is said regarding the solution techniques.

All of the above references are applicable only to systems that are described by ordinary differential equations, in contrast with the calculus of variations which also handles problems described by partial differential equations. Since many of the problems of mathematical physics involve partial differential equations, an extension of PMP to include this class of problems is the next logical development.

Some work has already been done, for example, Barnes (1971) and Komkov (1972). A survey of these "distributed parameter systems"--see Section 1.0--is presented in Wang (1968).

2.2.3 Method of Steepest Ascent/Descent

This method is frequently cited in the literature for trajectory optimization, and occasionally in references related to optimal structures. When the method of quasilinearization converged for the dissertation example problems, there was no need to investigate other methods such as the method of steepest ascent. Consequently, little is said about it. According to the references, it is applied in a straightforward manner. Furthermore, the example problem solutions presented seem to be real problems and not academically simple. The following four papers treat the method in general with trajectory optimization applications: Bryson and Denham (1962, 1964), Bryson et al. (1963), and Hanson (1968). In Haug et al. (1969), the method of steepest ascent is derived in detail, completely discussed, and compared to the maximum principle. Several structural optimization problems are then solved by the method of steepest ascent. Although no exciting results are obtained, use of the method is clearly illustrated by the applications to realistic structural problems.

2.2.4 Transition Matrix: Aeroelasticity Problems

For the past few years a group at Stanford University has studied the optimization of structures subject to dynamic or aerodynamic constraints. The general problem of their interest is that of minimizing

the weight of a given structure for specified eigenvalue, subject to inequality constraints on control.

Three types of solution techniques are used once the necessary conditions for minimum weight are determined. Exact solutions are obtained for most of the problems because they are so simple that analytical methods are applicable. More complicated problems are solved by a "transition matrix" method described in Bryson and Ho (1969). On the basis of convergence difficulties reported in the references, this method should be used with caution. Results have been obtained only for very simple problems. However, these results are corroborated by data obtained from a discrete approximation method. Five papers that are representative of this work are McIntosh and Eastep (1968), Ashley and McIntosh (1968), McIntosh et al. (1969), Ashley et al. (1970), and Weisshaar (1970).

2.2.5 Miscellaneous Methods

The preceding sections have briefly outlined the methods of structural optimization most frequently encountered in the literature. Appropriate, representative references have also been given. Not all methods are listed; while some are omitted for not being generally useful, others are omitted for not being generally used. Two examples of the latter are the "modified quasilinearization" and "sequential gradient-restoration" algorithms described in Miele et al. (1972) and in Hennig and Miele (1972). At some later time these methods may be acknowledged as major methods that are applicable to many different or important problems, but for now they are mentioned only in passing.

2.3 Methods: Discrete Systems

Discrete systems are described by a set of discrete constants instead of the set of functions associated with continuous systems. The classic example of a discrete system is a pin-connected truss, where the state x_i and control u_i are the stress and cross-sectional area, respectively, for each member i. A discrete system also arises in the approximation of a continuous system.

Several references that present a good discussion of general methods applied to discrete systems are available. Most of these exist in the form of an edited collection of papers by various authors on the topics of their acknowledged expertise. Four such publications are Gellatly (1970), Gellatly and Berke (1971), Pope and Schmidt (1971), and Gallagher and Zienkiewicz (1973). Another report, Melosh and Luik (1967), provides a good exposition of the difficulties associated with the analysis portion of least weight structural design. It also contains a brief comparison of various mathematical programming methods.

McNeill (1971) is the last reference to be cited in the section on general methods for the optimization of discrete systems. Minimum weight design of general structures is treated in a mathematically precise formulation. Legendre's necessary condition is combined with the concepts of convex functions and sets to derive the necessary and sufficient conditions for global optimality. Fully stressed designs and constraints to eigenvalues are also discussed. In summary, this paper provides a good example of the general mathematical problem that must be solved in the optimization of discrete systems.

While certain variational methods may be applied to discrete systems, the most frequently used technique is mathematical programming. In the following sections, this method and other major methods are discussed and representative references cited.

2.3.0 Mathematical Programming

The general method of mathematical programming is discussed in section 3.2 of the dissertation, and the solution of an example problem using this method is detailed in Chapter VI. In the literature related to this subject, a very readable textbook is available--Fox (1971). This book complements the theory with numerous discussions pertaining to numerical techniques and methods that can be employed to overcome certain difficulties that may arise. Although it does contain flowcharts of several algorithms, there are few specific examples given. For a discussion of the general theory, two alternatives to this book exist in the form of papers: Schmidt (1966, 1968). The first is written in a conversational style, contains no mathematics, and is intended to provide only a general description of the subject. The latter paper is theoretical in content.

An excellent application to a realistic problem is to be found in Stroud et al. (1971). Ths paper contains little discussion of the method itself, but does demonstrate an application that allows a conceptual visualization of the solution. The approach is to assume the solution to be a linear combination of specified functions, and to choose the weighting coefficients to minimize the cost. Mathematical programming is employed to determine the optimal set of coefficients.

This approach resembles Galerkin's method, and though not mathematically rigorous, it may provide a useful approximation to large, unwieldy problems.

2.3.1 Discrete Solution Approximations

In the previous section a paper is cited that contains an approximate solution obtained by Galerkin's method. The use of the Galerkin or Rayleigh-Ritz approximate solution techniques is sufficiently widespread to be considered a general method. For both methods, the solution is assumed to be a linear combination of the solution to the linear part of the governing equation and a set of prescribed functions. This approximate solution does not satisfy the given equation exactly but produces some residual function. A cost function that depends upon the residual is then minimized with respect to the unknown coefficients. The two weighted residual methods mentioned above have different cost functions, but the methods are identical for linear equations--see Cunningham (1958, p. 158).

The advantage to using these methods is that after assuming the particular form of solution, the problem of solving for the weighting coefficients may be much simpler than the original problem. In the case of Stroud et al. (1971), the coefficients were obtained by mathematical programming techniques. However, the weakness of the method is the restricted function space of possible solutions. With the coefficents obtained by these methods the resulting solution is the best approximation that is possible from the set of solution functions

prescribed. There is no guarantee that the approximation even resembles the true solution.

The flutter of a panel is solved using Galerkin's method in Plaut (1971a). No general developments are presented and the assumed solution functions are trivially simple. However, this paper does provide an application of the method to obtain an approximate solution to a very difficult optimization problem involving the stability of a nonconservative system. A similar problem is treated in a more theoretical manner in Plaut (1971b) using a modified Rayleigh-Ritz method. "Segmentwise-constant" control functions are assumed also; this particular approximation is discussed with more detail in the following section. Additional nonconservative problems are treated in Leipholz (1972), applying Galerkin's approximate solution to the energy method. Several simple examples are included.

Nonconservative elastic stability problems of elastic continua are treated in Prasad and Herrmann (1969) using adjoint systems. This approach is more realistic than the segmentwise-constant control assumption described in the following section. Solutions for the state and adjoint system are assumed, such that approximation process resembles the Rayleigh-Ritz method. However, only a single type of nonconservative system is considered. Extension to several other types of nonconservative elastic continua problems is given in Dubey (1970). Variational equations corresponding to both the Galerkin and Rayleigh-Ritz methods are derived. Furthermore, the condition for equivalence of the two methods is shown to be that the admissible velocity field must

satisfy a natural boundary condition over that portion of the body's surface where tractions are prescribed.

2.3.2 Segmentwise-Constant Approximations

The definitive characteristic of this method is approximating the structural system by a number of discrete segments, where within each segment the control function has a constant value. In general, the constant value of the control differs from segment to segment. For the many papers on this method that have been published, the procedure is the same. An optimality condition (necessary in all cases but also sufficient in some) or cost functional is derived for the continuous system. After defining the segmentwise-constant approximation, the condition or functional is reformulated in terms of the discrete parameters. Most of the papers use so few elements that solving for the discrete values of the control parameter poses no difficulties. Although this method does simplify the mathematical problem to be solved, the crudeness of the approximation is not appealing. Five papers which treat a variety of problems using this approximation are cited below.

Minimum weight of sandwich structures subject to static loads is discussed in Sheu and Prager (1968a). In Sheu (1968) the same type of structure is considered. It differs from the first problem by requiring point masses to be supported such that the total structure has a prescribed fundamental frequency of free vibration. Icerman (1969) treats the problem of elastic structures subject to a concentrated load of harmonically varying amplitude. The minimum weight design is obtained

subject to a compliance constraint related to the applied load, and which is effectively a boundary condition on displacement at the point of application. A truss problem is also included.

The concept of a compliance constraint is pursued further in Chern and Prager (1970). The minimum weight design for sandwich construction beams under alternative loads is found, subject to this type of constraint. The paper uses up to eight segments, thereby obtaining a more realistic approximation to the continuous problem. Minimum weight design of elastic structures subject to body forces and a prescribed deflection is discussed in Chern (1971a). This investigation is notable in that it considers applied loads that are functions of the design functions.

2.3.3 Complex Structures with Frequency Constraints

On the basis of useful application, perhaps the most important class of discrete structural optimization problems is the minimum weight design of complex structures subject to natural frequency constraints. Since most real structures are built with many structural elements of various types, and are not realistically described by any single type, this approach is more appropriate from the aspect of modeling the structure. Furthermore, many structures must be designed to avoid certain natural frequencies because of resonance or self-induced oscillations; this situation indicates that the natural frequency constraint is also appropriate.

Many different solution schemes have been developed which are usually based upon general mathematical programming techniques.

Typically, a design is iteratively altered to minimize the weight with a subsequent increase in frequency until a constraint is violated. At that point the design process uses an iteration which simultaneously reduces both weight and frequency. These two processes are repeated sequentially until no further weight reduction is possible.

Although circumstances may require the use of many elements, the number of them may itself be a critical factor. Some of the schemes require a matrix inversion as part of the eigenvalue problem solution associated with the frequency constraint. If the number of elements becomes too large, the size of the matrix to be inverted likewise becomes excessively large. When that occurs the matrix inversion can require excessive amounts of computer time. Another possible difficulty is that the inverse matrix itself is not sufficiently accurate, such that the subsequent calculations are not acceptable. However, for structures such as reinforced shells composed of different types of structural elements, this method may be the most applicable.

Many papers have been published pertaining to this class of structural optimization problem. Because the method is inherently oriented towards applications, the references are cited in chronological order without additional comments. Interested readers are referred to: Turner (1967), Zarghamee (1968), Turner (1969), De Silva (1969), Rubin (1970), Fox and Kapoor (1970), McCart et al. (1970), and Rudisill and Bhatia (1971).

2.3.4 Finite Element Approximations

There is an unfortunate ambiguity to the label "finite elements" that occurs because these words are used to describe two completely different entities. In papers cited in the preceding section they are used to indicate the discrete structural elements of finite dimensions which comprise the complex structure. The analysis of such systems of structural elements has been accomplished by ordinary matrix methods during the last three decades. However, during the past decade another method has been developed and named "the finite element method."

In this method a continuum is divided into small, finite elements over which a particular form of approximation of either the displacement and/or force is assumed. A number of nodes common to one or more element is prescribed; continuity is required to exist at these nodes but not necessarily elsewhere. An equilibrium equation is derived for each element, and then all of the individual equations are combined into a single equilibrium equation for the entire system. The resulting equation is a linear algebraic equation whose unknowns are displacements and/or forces at the nodes. Once the matrix equation is inverted, the nodal displacements and/or forces are used with the assumed approximation form to describe the state of the structure throughout each and every element, and hence the system. Hereafter this method is referred to as the "finite element method."

The most frequent application of the finite element method is to problems having complicated loads, geometry, and response. Generally speaking, the method is employed wherever the physical system is too

complex to be described adequately by a single differential equation and boundary conditions. For a complete theoretical development of the finite element method and numerous examples, see Zienkiewicz (1971).

With respect to structural optimization the method is employed to simplify the problem to be solved. Very little has been published on this subject, but the papers available cover a wide spectrum of techniques. For example, Dupuis (1971) combines the finite element and segmentwise-constant methods as applied to minimum weight beam design. A similar application to column buckling is contained in Simitses et al. (1973). Another paper, Wu (1973), is a study of two classical nonconservative stability problems. Although adapted to stability considerations, this presentation is the best exposition available in the open literature.

In Chapter VI a minimum deflection beam problem is solved with the combined methods of finite elements and mathematical programming.

2.4 Closure

In the preceding sections of this chapter, general problem types and methods are discussed. Only those methods that appear to have attained some standard of acceptance are presented. It must be acknowledged that other areas of important study exist but are perhaps overlooked as not being pertinent to the general subject area of the dissertation. As an example, Dorn et al. (1964) treats the optimal layout of trusses--an important subject but not related to the general problem to be considered in this dissertation. In addition only elastic structures have been considered although there are numerous

publications on optimal design of inelastic structures. References that are representative of this subject are: Drucker and Shield (1957a, 1957b), Hu and Shield (1961), Shield (1963), Prager and Shield (1967), and Mayeda and Prager (1967).

On considering the various references mentioned above it would appear that there are two possible pitfalls in structural optimization that should be avoided. The first is the confusing of method of optimization with the solution techniques employed to obtain a solution to the resulting TPBVP. In order to avoid possible errors the two should be dealt with independently, unless it is clearly advantageous to relate one to the other. Besides this it must be recognized that any solution obtained is "optimal" only with respect to the given conditions of the particular problem. Any change in the problem statement invalidates the applicability of that solution. The change may lead to a more desirable solution, but the original solution is no less valid. Simitses (1973) is an example where this situation is not acknowledged. In this paper the thickness of a thin reinforced circular plate of specified weight and diameter is determined such that the average deflection due to a uniform load is minimized. An earlier paper which did not include stiffening is cited with the implication that the optimum solution for the unstiffened plate is not correct. The point made above is that both of these solutions are optimum under the respective conditions of the two problems. Neither solution is more, or less, valid than the other.

CHAPTER III

THEORETICAL DEVELOPMENT

3.0 Introduction

This chapter contains the development of two distinct methods used in the theory of optimal processes, into a more general method. The first section defines precisely the problem to be considered. This includes the necessary conditions for an optimal solution given by the calculus of variations. Several mathematical programming techniques are described in the second section along with a numerical algorithm called the gradient projection method. The application of this numerical method to the solution of the necessary conditions from Pontryagin's Maximum Principle (PMP) is detailed in Section 3.3. Results of this approach are shown to be consistent with the necessary conditions, given in Section 3.1; these results provide a clarifying insight to the mathematical processes entailed in the maximum principle, and an explicit formulation for the Lagrangian multiplier functions. This explicit formulation is used in the next section to show the necessary conditions may then be regarded as an algorithm. The final section contains a brief summary of solution methods.

The main theoretical development of the dissertation is contained in the first three sections. It is well known that the problems encountered in the calculus of variations are equivalent to the optimization of

a functional (in the sense of mathematical programming problems) under certain restrictions upon the variations. A good exposition of this is available in Luenberger (1969). With this equivalence in mind, it is noted that the PMP is itself worded as a constrained optimization problem. When treated with what is normally regarded as a numerical method, the gradient projection method, an explicit formulation of the attendant Lagrangian multipliers is obtained. This form satisfies all of the calculus of variation necessary conditions and allows one to use them in a most straightforward fashion. As a result, these necessary conditions may be directly used in the form of an algorithm to obtain a solution. Furthermore, it is believed that treating the PMP as a mathematical programming problem in conjunction with the gradient projection method helps to explain the effect of combined control-state constraints upon the maximum principle.

3.1 Problem Statement and Necessary Conditions

A general problem which represents a large class of structural optimization problems is treated in the sequel. The functional

$$J = \int_0^{t_F} L_0(\underline{x},\underline{u})\ dt \tag{3.1.1}$$

is to be minimized with respect to the control $\underline{u}(t)$ where the state $\underline{x}(t)$ must satisfy certain boundary conditions and a differential constraint; in addition, an inequality constraint involving both the state and control must be satisfied. For

$$\underline{u}^T(t) = [u_1(t)\ u_2(t)\ \ldots\ u_m(t)] \qquad (3.1.2)$$

$$\underline{x}^T(t) = [x_1(t)\ x_2(t)\ \ldots\ x_n(t)] \qquad (3.1.3)$$

the subsidiary conditions to minimizing the cost function J are:

$$\dot{\underline{x}} = \underline{f}(\underline{x},\underline{u}) \qquad (3.1.4)$$

Specified Boundary Conditions on $\underline{x}(t)$ (3.1.5)

$$\phi_\ell(x,u) \leq 0 \qquad \ell = 1,\ldots,q \qquad (3.1.6)$$

Terminal time t_F is considered to be constant; allowing it to be unspecified requires only a slight modification to the following derivation.

This problem is a particular form of a very general one treated by Hestenes (1966). His results are a set of necessary conditions which must be satisfied by the optimal solution and include the maximum principle. To obtain the necessary conditions, the inequality constraints are converted to equality constraints in the manner of Valentine (1937). These constraints and the differential constraints are then adjoined to the cost function via Lagrangian multiplier functions $\mu_\ell(t)$ and $p_i(t)$ respectively.

$$\phi_\ell(\underline{x},\underline{u}) + s_\ell^2(t) = 0$$

where the slack variables $s_\ell(t)$ are defined* such that

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*The symbol "$\triangleq$" denotes "is defined by."

$$s_\ell(t) \triangleq [-\phi_\ell(\underline{x},\underline{u})]^{\frac{1}{2}} \geq 0$$

$$J = \int_0^{t_F} L_0(\underline{x},\underline{u})\,dt + \int_0^{t_F} \underline{p}^T(t)\,[\dot{\underline{x}} - \underline{f}(\underline{x},\underline{u})]\,dt +$$

$$+ \int_0^{t_F} \mu_\ell(t)\,[\mu_\ell(\underline{x},\underline{u}) + s_\ell^2(t)]\,dt$$

Implied summation convention is used whenever a vector formulation leads to possible ambiguities in later developments. Integrating the second integral by parts gives a result that leads to the variational Hamiltonian.

$$J = \underline{p}^T\underline{x}\Big|_0^{t_F} + \int_0^{t_F} [L_0 - \underline{p}^T\underline{f} + \mu_\ell\phi_\ell - \underline{x}^T\dot{\underline{p}} + \mu_\ell s_\ell^2]\,dt$$

Define:

$$H(\underline{x},\underline{u},\underline{p}) \triangleq L_0(\underline{x},\underline{u}) - \underline{p}^T(t)\,\underline{f}(\underline{x},\underline{u}) \tag{3.1.7}$$

the variational Hamiltonian, and H^* which will include terms arising from the inequality constraint.

$$H^* = -H - \mu_\ell\phi_\ell$$

or

$$H^* = \underline{p}^T(t)\,\underline{f}(\underline{x},\underline{u}) - L_0(\underline{x},\underline{u}) - \mu_\ell(t)\,\phi_\ell(\underline{x},\underline{u})$$

Hence

$$J = \underline{p}^T\underline{x}\Big|_0^{t_F} - \int_0^{t_F} [H^* + \underline{x}^T\dot{\underline{p}} - \mu_\ell s_\ell^2]\,dt$$

With the exception of the maximum principle, all of Hestenes' necessary conditions are obtained from the requirement that the first variation of the cost function vanish. In the following, "$\delta\underline{x}$" designates

"the variation of $\underline{x}$"; a subscript vector designates the partial derivative with respect to that vector, with the result itself a column vector. Thus,

$$\delta J = \underline{p}^T \delta \underline{x} \Big|_0^{t_F} - \int_0^{t_F} [\delta \underline{x}^T (H^*_{\underline{x}} - \dot{\underline{p}}) +$$

$$+ \delta \underline{u}^T H^*_{\underline{u}} - \mu_\ell 2 s_\ell \delta s_\ell] \, dt = 0$$

To derive the PMP requires an extensive mathematical development and is not included since it contributes nothing to the present discussion. However, the necessary conditions are listed in order to be available for later reference.

$$\dot{\underline{x}} = H^*_{\underline{p}} = \underline{f}(\underline{x}, \underline{u})$$

$$\dot{\underline{p}} = -H^*_{\underline{x}}$$

$$\left.\begin{array}{ll} 0 = p_i \delta x_i & , \quad t = 0 \\ 0 = p_i \delta x_i & , \quad t = t_F \\ \text{Specified Boundary Conditions on } x_i(t) & \end{array}\right\}$$

$$0 = H^*_{\underline{u}}$$

$$0 = \mu_\ell(t) \, \phi_\ell(\underline{x}, \underline{u}) \quad , \quad \mu_\ell(t) \geq 0$$

$$H(\underline{x}_{OPT}, \underline{u}_{OPT}, \underline{p}) \leq H(\underline{x}_{OPT}, \underline{u}, \underline{p})$$

The optimal solution must satisfy these six conditions together with the inequality constraint (3.1.6).

The PMP states that along the optimal trajectory, each instant of time t, state $\underline{x}_{OPT}(t)$ and adjoint state $\underline{p}(t)$, treated as fixed, the optimal control $\underline{u}_{OPT}(t)$ is that admissible control which minimizes the variational Hamiltonian. In the present context, admissibility requires that $\underline{u}(t)$ be piecewise continuous, the set of admissible controls being denoted by Ω. Hence the PMP indicates that

$$\underline{u}_{OPT}(t) = \underset{\underline{u}\,\varepsilon\,\Omega}{\text{ARGMIN}}\ [H(\underline{x}_{OPT}, \underline{u}\ , \underline{p})] \tag{3.1.8}$$

Notice that the necessary conditions suggest nothing about how a solution is obtained, but merely indicate certain functional relationships that must be satisfied. However, equation (3.1.8) seems to intimate that solution of the necessary condition of PMP involves a mathematical programming problem.

3.2 Mathematical Programming: Gradient Projection Method

Having shown that the PMP from the calculus of variations approach to an optimization problem may perhaps be related to a mathematical programming problem, the latter will be discussed in general terms. Consider a nonlinearly constrained optimization problem

$$\underline{x}_{OPT} = \underset{\underline{x}\,\varepsilon\,\Omega}{\text{ARGMIN}}\ [F(x)]$$

subject to

$$g_j(\underline{x}) \leq 0 \qquad j = 1,\ldots,m$$

where Ω denotes the set of admissible state components x_i, $i = 1,\ldots,n$, and to be admissible requires only the satisfaction of the m inequalities.

Necessary conditions which $\underline{x}_{OPT}$ must satisfy are given in the Kuhn-Tucker theorem:

(i) constraints are satisfied $\qquad g_j(\underline{x}_{OPT}) \leq 0$

(ii) multipliers exist such that $\qquad \lambda_j \geq 0$

and for all $j = 1,\ldots,m$ $\qquad \lambda_j g_j(\underline{x}_{OPT}) = 0$

(iii) and $\qquad \nabla_{\underline{x}} F(\underline{x}_{OPT}) + \sum_{j=1}^{m} \lambda_j \nabla_{\underline{x}} g_j(\underline{x}_{OPT}) = 0$

Observe that if I_A denotes the set of indices associated with active constraints, the first two conditions may be written as

$$j \in I_A \rightarrow g_j(\underline{x}) = 0 \quad \text{and} \quad \lambda_j > 0$$

$$j \notin I_A \rightarrow g_j(\underline{x}) < 0 \quad \text{and} \quad \lambda_j = 0$$

Fox (1971, pp. 168-176) presents a very readable proof of this theorem; a more mathematical proof using vector space concepts is available in Luenberger (1969).

Many methods for obtaining a numerical solution to the nonlinear programming problem described by the first two equations of this section have been developed. The gradient projection method by Rosen (1960) is used frequently in structural optimization. Basic to the method is the orthogonal projection of the cost function gradient onto a subspace defined by the normal vectors of the active constraints. An inherent part of the algorithm is the concept of a "feasible," "usable" direction. Any direction $\hat{d}$ is feasible if an increment $\underline{x}$ in that direction improves the cost function, i.e., decreases $F(\underline{x})$. Direction $\hat{d}$ is said to be usable if it also satisfies the constraints. As long as a feasible,

usable direction exists, the cost function may be improved. A constrained optimal solution $\underline{x}_{OPT}$ occurs at that point where no feasible direction is also usable, i.e., any attempt to improve the cost violates a constraint. In Appendix B these concepts are used in a concise proof of the Kuhn-Tucker conditions.

Fox (1971) derives the matrix P which projects the cost function gradient into the subspace defined by vectors normal to the active constraints. This is equivalent to subtracting all components parallel to vectors that are normal to surfaces of active constraints from the negative gradient of the cost function. Recalling the definition of set I_A, consider r constraints to be active such that

$$I_A = \{\alpha_1, \alpha_2, \ldots, \alpha_r\}$$

Define a vector whose elements are the corresponding nonzero Lagrangian multipliers, and another vector whose elements are the active constraint functions

$$\underline{\lambda}^T = [\lambda_{\alpha_1} \; \lambda_{\alpha_2} \; \ldots \; \lambda_{\alpha_r}]$$

$$\underline{N}^T = [g_{\alpha_1} \; g_{\alpha_2} \; \ldots \; g_{\alpha_r}]$$

From the $\underline{N}$ vector, a matrix N is introduced, each column of which is the gradient of an active constraint. Hence, N is an $(n \times r)$ matrix where

$$\left.\begin{aligned} N &= \underline{N}^T_{\underline{x}} = [N_{ij}] \\ &\text{and} \\ N_{ij} &= \frac{\partial N_j}{\partial x_i} = \frac{\partial g_{\alpha j}}{\partial x_i} \end{aligned}\right\} \quad \begin{aligned} i &= 1, \ldots, n \\ \\ j &= 1, \ldots, r \end{aligned} \tag{3.2.1}$$

With these definitions of $\underline{\lambda}$ and N, the third Kuhn-Tucker condition can be written as

$$\nabla_{\underline{x}} F(\underline{x}) + N\,\underline{\lambda} = 0$$

At any feasible point $\underline{x}$ where $g_j(\underline{x}) \leq 0$, the direction which best improves the cost function is the negative gradient of the cost. If those directions which lead to constraint violations are subtracted from $-\nabla_{\underline{x}} F(\underline{x})$, the projection matrix P is obtained. Directions causing a constraint violation are specified by the gradients of active constraints, i.e., the columns of $\underline{N}^T_{\underline{x}}$. What is required of $\hat{S}$, the projection of the gradient, is that

$$\hat{S} = (-\nabla_{\underline{x}} F(\underline{x})) - \underline{N}^T_{\underline{x}}\,\underline{\lambda} \tag{3.2.2}$$

where $\underline{\lambda}$ are scalar coefficients to be determined such that $\hat{S}$ is orthogonal to each column of $\underline{N}^T_{\underline{x}}$, or

$$(\underline{N}^T_{\underline{x}})^T\,\hat{S} = 0$$

When the matrix equivalent to $\underline{N}^T_{\underline{x}}$ is used together with the $\hat{S}$ expression (3.2.2), the result is

$$N^T(-\nabla_{\underline{x}} F(\underline{x}) - N\,\underline{\lambda}) = 0$$

such that the $\underline{\lambda}$ which satisfies this orthogonality condition is:

$$\underline{\lambda} = -\,(N^T N)^{-1}\,N^T(\nabla_{\underline{x}} F(\underline{x})) \tag{3.2.3}$$

Unless the active boundary surface normals $\nabla_{\underline{x}} g_j(\underline{x})$ are linearly dependent, the matrix $(N^T N)$ is nonsingular. Conversely, if this matrix is

singular the active constraints are not linearly independent; however, this is not a condition encountered in most real cases.

Substitution of the $\underline{\lambda}$ expression into the $\hat{S}$ equation leads directly to the projection matrix P:

$$\hat{S} = -P \nabla_{\underline{x}} F(\underline{x})$$

where

$$P = I - N(N^T N)^{-1} N^T \tag{3.2.4}$$

where I is the identity matrix. The direction $\hat{S}$ which best improves the cost is given in terms of P, where P and N are given by (3.2.4) and (3.2.1). If no constraints are active at a point $\underline{x}$, then N is a null matrix, P reduces to an identity matrix, and the direction of best improvement is coincident with the direction of steepest descent.

In the algorithm associated with this method the starting point must be a feasible point where $g_j(\underline{x}) \leq 0$ for all $j = 1,\ldots,m$. The design then proceeds in the $\hat{S}$ direction until the solution is satisfied to within a specified position tolerance ε. Necessary conditions generally programmed in a computer program are:

$$|S_j| \leq \varepsilon, \quad i = 1,\ldots,n$$

$$\lambda_j > 0, \quad j \in I_A$$

$$\lambda_j = 0, \quad j \notin I_A$$

It is readily seen that for $\hat{S}$, P, and $\underline{\lambda}$ defined as above, these are completely equivalent to the Kuhn-Tucker conditions.

3.3 Gradient Projection Method Applied to the Maximum Principle

Based upon the preceding discussion, the similarity between PMP and the mathematical programming problem can be discussed. The maximum principle states that the optimal control $\underline{u}_{OPT}$ minimizes the variational Hamiltonian with respect to all admissible $\underline{u}$. Or, at each time $0 \leq t \leq t_F$, $\underline{u}_{OPT}$ minimizes $H(\underline{x},\underline{u},\underline{p})$ with respect to $\underline{u}$ for given $\underline{x}$ and $\underline{p}$ and where $\underline{u}_{OPT}$ is subject to constraints $\phi_\ell(\underline{x},\underline{u}) \leq 0$, $\ell = 1,\ldots,q$. Treating this as a mathematical programming problem, the following correspondences are recognized

$$\underline{x} \sim \underline{u}$$

$$F(\underline{x}) \sim H(\underline{x},\underline{u},\underline{p})$$

$$g_j(\underline{x}) \sim \phi_\ell(\underline{x},\underline{u})$$

$$\lambda_j \sim \mu_\ell(t)$$

$$\nabla_{\underline{x}} F(\underline{x}) \sim H_{\underline{u}}$$

$$\hat{S} \sim H_{\underline{u}} \qquad \text{plus constraints}$$

Continuing to identify corresponding quantities, at each time t, let I_A denote the set of active constraints, taken to be r in number.

$$I_A = \{\alpha_1, \alpha_2, \ldots, \alpha_r\}$$

Then

$$\underline{N}^T \sim \underline{\phi}^T = [\phi_{\alpha_1} \; \phi_{\alpha_2} \; \cdots \; \phi_{\alpha_r}]$$

$$\underline{\lambda}^T \sim \underline{\mu}^T = [\mu_{\alpha_1}(t) \; \mu_{\alpha_2}(t) \; \ldots \; \mu_{\alpha_r}(t)]$$

$$N \sim \underline{\phi}_{\underline{u}}^T = [\phi_{ij}] = \left[\frac{\partial \phi_{\alpha_j}}{\partial x_j}\right] \qquad \begin{array}{l} j = 1,\ldots,n \\ j = 1,\ldots,m \end{array}$$

Furthermore, define $(H_{\underline{u}})_P$ to be the gradient of H with respect to $\underline{u}$ where all components that cause a constraint violation have been removed. Since projection matrix P removes cost function gradient components that lead to constraint violations, consider its use in the maximum principle. With the correspondent to N identified as $\underline{\phi}_{\underline{u}}^T$, then

$$P = I - \underline{\phi}_{\underline{u}}^T (\underline{\phi}_{\underline{u}^T} \underline{\phi}_{\underline{u}}^T)^{-1} \underline{\phi}_{\underline{u}^T} \tag{3.3.1}$$

and

$$(H_{\underline{u}})_P = P\, H_{\underline{u}} \tag{3.3.2}$$

From the Kuhn-Tucker conditions, this implies that along the optimal trajectory $(t, \underline{x}_{OPT}, \underline{u}_{OPT})$

$$\hat{S} = -(H_{\underline{u}})_P = 0 \tag{3.3.3}$$

Similarly, at each time

$$\underline{\mu}(t) - (\underline{\phi}_{\underline{u}^T} \underline{\phi}_{\underline{u}}^T)^{-1} \underline{\phi}_{\underline{u}^T} H_{\underline{u}} \tag{3.3.4}$$

from which it follows

$$(\underline{\phi}_{\underline{u}^T} \underline{\phi}_{\underline{u}}^T)\underline{\mu} + \underline{\phi}_{\underline{u}^T} H_{\underline{u}} = 0$$

$$\underline{\phi}_{\underline{u}}^T \underline{\mu} + H_{\underline{u}} = 0$$

$$(H + \underline{\mu}^T \underline{\phi})_{\underline{u}} = 0 \tag{3.3.5}$$

$$(\ -H^{*}\ )_{\underline{u}} = 0$$

Or,

$$H^{*}_{\underline{u}} = 0 \tag{3.3.6}$$

Hence the control law from Hestenes' necessary conditions can be derived from the PMP condition by treating it as a nonlinearly constrained mathematical programming problem. While using the gradient projection method in the derivation, it is seen that equation (3.3.5) is equivalent to the third Kuhn-Tucker condition. The second Kuhn-Tucker condition is identical to Hestenes' necessary condition on the Lagrangian multipliers used to adjoin the inequality constraints to the cost function. Satisfaction of the inequality is implied by requiring the first Kuhn-Tucker condition to be fulfilled, where

$$\lambda_j \geq 0 \quad \rightarrow \quad \mu_\ell(t) \geq 0 \tag{3.3.7}$$

$$\lambda_j g_j(\underline{x}) = 0 \rightarrow \mu_\ell(t)\phi_\ell(\underline{x},\underline{u}) = 0 \tag{3.3.8}$$

$$g_j(\underline{x}) \leq 0 \quad \rightarrow \quad \phi_\ell(\underline{x},\underline{u}) \leq 0 \tag{3.3.9}$$

Thus by treating the solution of the necessary conditions of the maximum principle as a programming problem with inequality constraints, using the gradient projection matrix, and by requiring satisfaction of the Kuhn-Tucker conditions, an explicit formula for Hestenes' Lagrangian multiplier functions has been derived. It is further demonstrated that with the $\mu_\ell(t)$ so defined satisfaction of the extremum control law condition is implied. However, before this treatment can be accepted as valid, it must also be shown that the system of canonical differential equations is unchanged.

Consider the state system equations

$$\dot{\underline{x}} = H^*_{\underline{p}} = \underline{f}(\underline{x},\underline{u})$$

where

$$H^* = \underline{p}^T(t)\underline{f}(\underline{x},\underline{u}) - L_0(\underline{x},\underline{u}) - \underline{\mu}^T(t)\underline{\phi}(\underline{x},\underline{u})$$

It is obvious that the explicit form of $\mu(t)$ has absolutely no effect upon the state system equation expressed in canonical form.

Demonstrating that the adjoint system equation is unchanged requires the method described by Bryson et al. (1964). Consider the general problem of Section 3.1 again, but with only the differential constraints adjoined to the cost function, i.e.,

$$\underset{\underline{u}}{\text{Min}} \{J = \underline{p}^T\underline{x}\Big|_0^{t_F} + \int_0^{t_F} (H - \underline{x}^T\dot{\underline{p}})dt\} \tag{3.3.10}$$

$$\text{subject to:} \quad \phi_\ell(\underline{x},\underline{u}) \le 0 \qquad \ell = 1,\ldots,q \tag{3.1.6}$$

where $$H(\underline{x},\underline{u},\underline{p}) = L_0(\underline{x},\underline{u}) - \underline{p}^T(t)\underline{f}(\underline{x},\underline{u}) \tag{3.1.7}$$

and $$\dot{\underline{x}} = \underline{f}(\underline{x},\underline{u}) \tag{3.1.4}$$

Again let I_A denote the set of indices associated with r active constraints at any time t

$$I_A = \{\alpha_1, \alpha_2, \ldots,\alpha_r\}$$

$$\phi_\ell(\underline{x},\underline{u}) < 0 \quad , \quad \ell \notin I_A \rightarrow \mu_\ell(t) = 0$$

$$\phi_\ell(\underline{x},\underline{u}) = 0 \quad , \quad \ell \in I_A \rightarrow \mu_\ell(t) > 0$$

and $\underline{\phi}$ is defined as before

$$\underline{\phi}^T = [\phi_{\alpha_1} \ \phi_{\alpha_2} \ \cdots \ \phi_{\alpha_r}]$$

The problem can then be thought of as minimizing (3.3.10) subject to

$$\underline{\phi}(\underline{x},\underline{u}) = 0$$

While on the constraint surfaces defined by this equation the variations in control $\delta \underline{u}(t)$ and state $\delta \underline{x}(t)$ are not independent but instead are related through the subsidiary requirement that

$$\delta \underline{\phi}(\underline{x},\underline{u}) = 0$$

or

$$\delta \underline{x}^T \underline{\phi}_{\underline{x}}^T (\underline{x},\underline{u}) + \delta \underline{u}^T \underline{\phi}_{\underline{u}}^T (\underline{x},\underline{u}) = 0 \qquad (3.3.11)$$

This imposes a restriction to the admissible variations. For cost function (3.3.10) to be a minimum, it is necessary that its first variation vanish, i.e.,

$$\delta J = \underline{p}^T \delta \underline{x} \Big|_0^{t_F} + \int_0^{t_F} \left[\delta \underline{x}^T H_{\underline{x}} + \delta \underline{u}^T H_{\underline{u}} - \delta \underline{x}^T \dot{\underline{p}} \right] dt = 0 \qquad (3.3.12)$$

It has already been shown that

$$H^*_{\underline{u}} = -(H + \underline{\mu}^T \underline{\phi})_{\underline{u}} = 0$$

which will be used to advantage shortly, after having added and subtracted the term $\delta \underline{u}^T (\underline{\mu}^T \underline{\phi})_{\underline{u}}$ from the integrand of (3.3.12).

$$0 = \underline{p}^T \delta \underline{x} \Big|_0^{t_F} + \int_0^{t_F} \Big[\delta \underline{x}^T (H_{\underline{x}} - \dot{\underline{p}}) + \delta \underline{u}^T H_{\underline{u}} + $$

$$+ \delta \underline{u}^T (\underline{\mu}^T \underline{\phi})_{\underline{u}} - \delta \underline{u}^T (\underline{\mu}^T \underline{\phi})_{\underline{u}} \Big] dt$$

Rearranging terms gives

$$0 = \underline{p}^T \delta \underline{x} \Big|_0^{t_F} + \int_0^{t_F} \Big[\delta \underline{x}^T (H_{\underline{x}} - \dot{\underline{p}}) + $$

$$+ \delta \underline{u}^T (H_{\underline{u}} + (\underline{\mu}^T \underline{\phi})_{\underline{u}}) - \delta \underline{u}^T (\underline{\mu}^T \underline{\phi})_{\underline{u}} \Big] dt$$

But,

$$(\underline{H}_{\underline{u}} + (\underline{\mu}^T \underline{\phi})_{\underline{u}} = (\underline{H} + \underline{\mu}^T \underline{\phi})_{\underline{u}} = (-H^*)_{\underline{u}} = 0$$

and

$$(\underline{\mu}^T \underline{\phi})_{\underline{u}} = \underline{\phi}_{\underline{u}}^T \; \underline{\mu}$$

Hence,

$$0 = \underline{p}^T \delta \underline{x} \Big|_0^{t_F} + \int_0^{t_F} \left[\delta \underline{x}^T (\underline{H}_{\underline{x}} - \dot{\underline{p}}) - \delta \underline{u}^T \underline{\phi}_{\underline{u}}^T \; \underline{\mu} \right] dt$$

It is here that the restrictions imposed by the active constraints are applied; from (3.3.11)

$$- \delta \underline{u}^T \; \underline{\phi}_{\underline{u}}^T = \delta \underline{x}^T \; \underline{\phi}_{\underline{x}}^T$$

such that

$$0 = \underline{p}^T \delta \underline{x} \Big|_0^{t_F} + \int_0^{t_F} \left[\delta \underline{x}^T (\underline{H}_{\underline{x}} - \dot{\underline{p}}) + \delta \underline{x}^T \; \underline{\phi}_{\underline{x}}^T \; \underline{\mu} \right] dt$$

$$0 = \underline{p}^T \delta \underline{x} \Big|_0^{t_F} + \int_0^{t_F} \left[\delta \underline{x}^T (\underline{H}_{\underline{x}} - \dot{\underline{p}} + \underline{\phi}_{\underline{x}}^T \; \underline{\mu}) \right] dt$$

Applying Euler's lemma, for arbitrary variations in the state which satisfies the constraints,

$$(\underline{H}_{\underline{x}} - \dot{\underline{p}} + \underline{\phi}_{\underline{x}}^T \; \underline{\mu}) = 0$$

which by the following manipulations is shown to be the adjoint system equation of Hestenes.

$$\dot{\underline{p}} = H_{\underline{x}} + (\underline{\mu}^T \underline{\phi})_{\underline{x}}$$

$$= - (-H - \underline{\mu}^T \underline{\phi})_{\underline{x}}$$

$$\dot{\underline{p}} = - H_{\underline{x}}^*$$

Thus, the explicit formulation for $\underline{\mu}(t)$ obtained by applying the gradient projection method to the PMP satisfies all the necessary conditions of Hestenes.

It may happen that in some cases the constraint upon control does not depend upon the state. It can be shown that the $\underline{\mu}(t)$ explicit formulation is equally valid in this instance. By examination of equations (3.3.1) through (3.3.9) it can be verified that all the necessary conditions except the adjoint system equation are satisfied. To demonstrate the latter, recall that when on a constraint boundary the first variation of both the cost functional and the constraint function must vanish. That is, for

$$\underline{\phi}(\underline{u}) = 0$$

both

$$\delta J = 0$$

and

$$\delta\underline{\phi} = \delta\underline{u}^T \, \underline{\phi}_{\underline{u}}^T = 0 \tag{3.3.13}$$

To derive the desired equivalence, the same term must be added and subtracted from the integrand of δJ as before, again arriving at the result

$$0 = \underline{p}^T \delta\underline{x}\Big|_0^{t_F} + \int_0^{t_F} \left[\delta\underline{x}^T (H_{\underline{x}} - \dot{\underline{p}}) - \delta\underline{u}^T \, \underline{\phi}_{\underline{u}}^T \, \underline{\mu}\right] dt$$

When the constraint variation (3.3.13) is introduced into this last equation, then by Euler's lemma

$$(H_{\underline{x}} - \dot{\underline{p}}) = 0$$

Since it was stipulated that $\underline{\phi}(u)$ is not a function of $\underline{x}$, the equation may be written

$$(H + \underline{\mu}^T \underline{\phi})_{\underline{x}} - \dot{\underline{p}} = 0$$

$$(\; - H^* \;)_{\underline{x}} - \dot{\underline{p}} = 0$$

$$\dot{\underline{p}} = H^*_{\underline{x}}$$

Thus, the expression for $\underline{\mu}(t)$ is valid when the constraint inequality depends only upon the control $\underline{u}(t)$.

3.4 Maximum Principle Algorithm

In the introduction to this chapter it was stated that the Lagrange type problem from the calculus of variations is equivalent to an ordinary mathematical programming problem based on the Kuhn-Tucker conditions. Furthermore, when inequality constraints are present the necessary conditions are equivalent to the Kuhn-Tucker conditions. It was demonstrated in the preceding section that if the PMP is itself treated as a mathematical programming problem, application of the gradient projection method provides an explicit solution for the Lagrangian multipliers associated with active constraints. This explicit solution for $\mu_\ell(t)$ also satisfies all of the other necessary conditions for an optimal solution. The ability to determine $\mu_\ell(t)$ explicitly in terms of parameters and functions that describe the problem suggests the possibility of converting the necessary condtions of an optimal solution into an algorithm for obtaining it.

Ensuing discussion of the algorithmic form of the necessary conditions contains the implicit assumption that all equations are valid along the optimal trajectory. It is further assumed that the problem under consideration is that one described in equations (3.1.1 - 3.1.6). The algorithm requires that $\underline{x}(t)$ and $\underline{p}(t)$ be known at each time $0 \leq t \leq t_F$ for which the solution procedure is as follows.

(i) Use PMP on the variational Hamiltonian to determine an optimal control $\underline{u}^*(t)$ independent of constraints.

$$\underline{u}^*(t) = \text{ARGMIN } [H(\underline{x},\underline{u},\underline{p})]$$

Evaluating the inequality constraints with $\underline{u} = \underline{u}^*$ reveals which of the $\ell = 1,\ldots,q$ constraints are active. Let r denote the number of active constraints and I_A the set of indices associated with them.

$$I_A = \{\alpha_1, \alpha_2, \ldots, \alpha_r\}$$

$$\phi_\ell(\underline{x},\underline{u}) = 0 \qquad \ell \in I_A$$

$$\phi_\ell(x,u) < 0 \qquad \ell \notin I_A$$

From this the vectors whose elements are the nonzero Lagrangian multipliers and corresponding constraint functions are defined, respectively, at the instant of time t.

$$\underline{\mu}^T(t) = [\mu_{\alpha_1} \ \mu_{\alpha_2} \ \cdots \ \mu_{\alpha_r}]$$

$$\underline{\phi}^T(\underline{x},\underline{u}) = [\phi_{\alpha_1} \ \phi_{\alpha_2} \ \cdots \phi_{\alpha_r}]$$

(ii) Having identified which of the q constraints are active, r components of $\underline{u}_{OPT}$ are specified by $\underline{\phi}(\underline{x},\underline{u}) = 0$. They may be solved by using the implicit function theorem, which requires

$\underline{\phi}_{\underline{u}}^T$ to be of rank r. This in turn requires the r constraints which are active at point $\underline{x}(t)$ to be linearly independent. To determine the remaining (m - r) components of $\underline{u}_{OPT}$ requires that $\underline{\mu}$ be known at time t, but

$$\underline{\mu}(t) = - (\underline{\phi}_{\underline{u}^T} \underline{\phi}_{\underline{u}}^T)^{-1} \underline{\phi}_{\underline{u}^T} H_{\underline{u}} .$$

This value of $\underline{\mu}$ is used to determine the "constrained" Hamiltonian,

$$H^* = - (H + \underline{\mu}^T \underline{\phi})$$

(iii) With the nonzero Lagrangian multipliers $\underline{\mu}$ known and H^* consequently defined, the remaining (m-r) unknown components of $\underline{u}_{OPT}$ are determined from the control law for the constrained system, i.e., from

$$H^*_{\underline{u}} = 0$$

Once $\underline{u}_{OPT}$ is completely known, the adjoint system equations are determined by

$$\dot{\underline{p}} = - H^*_{\underline{x}}$$

The process outlined above then allows $\underline{u}_{OPT}$ to be written as

$$\underline{u}_{OPT} = \underset{\underline{u}\,\varepsilon\,\Omega}{\text{ARGMIN}} \, [H(\underline{x}_{OPT}, \underline{u}, \underline{p})]$$

since the $\underline{u}$ obtained in this fashion satisfies $\phi_\ell(\underline{x}, \underline{u}) \leq 0$ which is the only requirement for being admissible. However it must be recalled that these equations are valid along the optimal trajectory; it remains to be shown that this algorithm may be employed in some manner to obtain that optimal trajectory and to demonstrate their satisfaction along it.

3.5 Solution Methods

Necessary conditions from the calculus of variations provides a Two Point Boundary Value Problem (TPBVP) to be solved, which is in general, nonlinear. For all but the most simple problems no analytical solution is possible and if any solution is to be obtained a computer must be used with some numerical method. A discussion of the available methods and their relative advantages/disadvantages is not included here due to the availability of such discussions in the literature, e.g., Bullock (1966). All of the methods involve some iterative scheme, and for optimal control can be separated into two general categories.

(i) Indirect methods. Schemes which require an intial guess of the state's solution: In these methods the starting point is an initial guess of the time history of the solution. The control associated with the solution is a subsequent calculation. Iteration continues until the state satisfies some criterion connoting convergence; the final control history at convergence is the optimal control.

(ii) Direct methods. Schemes which require an initial guess of the control function: The starting point for these methods is an initial guess of the control time history. For this class of methods the state associated with the control is a subsequent calculation. Iteration continues until the control satisfies some convergence criterion.

The method of quasilinearization was selected, based upon the success of Boykin and Sierakowski (1972) in applying it to a constrained

structural optimization problem. Excellent convergence for their problem, the capability to handle nonlinear systems, and the availability as an IBM SHARE program, ABS QUAS1, dictated its selection. In the application to the examples in Chapters IV and V the program required no modification. As a result, a detailed discussion of the method of quasilinearization is not included.

The problem discussed in preceding sections of this chapter falls into the general class of problems that QUAS1 handles, that is,

$$\dot{\underline{Y}} = \underline{g}(\underline{Y}, t)$$

with the boundary condition of the form

$$B_\ell \underline{Y}(0) + B_r \underline{Y}(t_F) + \underline{C}_Q = 0$$

where t_F, square matrices B_ℓ and B_r, and vector $\underline{C}_Q$, are specified, constant quantities. The specific form of B_ℓ, B_r and $\underline{C}_Q$ depend upon the given boundary conditions. As described in algorithm form

$$\dot{\underline{x}} = \underline{f}(\underline{x}, \underline{u}_{OPT}(\underline{x},\underline{p})) = \underline{G}_1(\underline{x},\underline{p})$$

$$\dot{\underline{p}} = -H^*_{\underline{x}}(\underline{x},\underline{u}_{OPT}(\underline{x},\underline{p})) = \underline{G}_2(\underline{x},\underline{p})$$

In terms of the general QUAS1 nomenclature,

$$\underline{Y} = \begin{bmatrix} \underline{x} \\ \\ \underline{p} \end{bmatrix} \qquad\qquad \underline{g}(\underline{Y}, t) = \begin{bmatrix} G_1(\underline{x},\underline{p}) \\ \\ G_2(\underline{x},\underline{p}) \end{bmatrix}$$

Boundary conditions are determined by those specified for the original system and by the necessary conditions outlined in the first section of this chapter.

In Chapter VI the problem treated by Boykin and Sierakowski (1972) is solved by the gradient projection method applied to a finite element

formulation for the description of the structural system. This is a method of the second kind mentioned above. Results of the two methods are compared.

CHAPTER IV

CONSTRAINED DESIGN OF A CANTILEVER BEAM
BENDING DUE TO ITS OWN WEIGHT

4.0 Introduction

A structural optimization problem has been selected for its simplicity and stated as an optimal control problem. The maximum principle is applied, giving a nonlinear TPBVP of the Mayer type. Among the earliest expository papers on the maximum principle, Rozonoér (1959) gives an excellent treatment to a similar type of problem; his technique is used to obtain both the variational Hamiltonian and adjoint variable boundary conditions. It is shown that no finite solution exists for the situation of unconstrained control. Numerical solutions for constrained control are obtained by the method of quasilinearization. Constraints include both geometric limitations to control as well as maximum stress limits that become mixed constraints depending upon both state and control variables.

4.1 Problem Statement

A cantilever beam of variable rectangular cross section is to be designed for minimum tip deflection due solely to its own weight. The material is specified to the extent that the modulus E and density ρ are constants. Length L is specified but the design variables, height h(x) and width w(x), may be chosen independently of each other, subject to

hard constraints upon the allowable dimensions. That is

$$\begin{aligned} a &\leq w(x) \leq c \\ b &\leq h(x) \leq d \end{aligned} \tag{4.1.1}$$

If $Y(x)$ denotes the deflection of the centerline, the problem is: given E, ρ, L, and the constraints, find $h(x)$ and $w(x)$ to minimize $Y(L)$. The particular form of differential constraints to be satisfied will be derived in the next section.

4.2 Structural System

Small deflections are assumed in order to use linear Bernoulli-Euler bending theory. Basic conventions assumed for this example are depicted in Figure 4.1; with these conventions the governing equation is derived using standard strength of materials considerations. The result is

$$EI_B(x)Y''(x) = M_B(x) \tag{4.2.1}$$

where

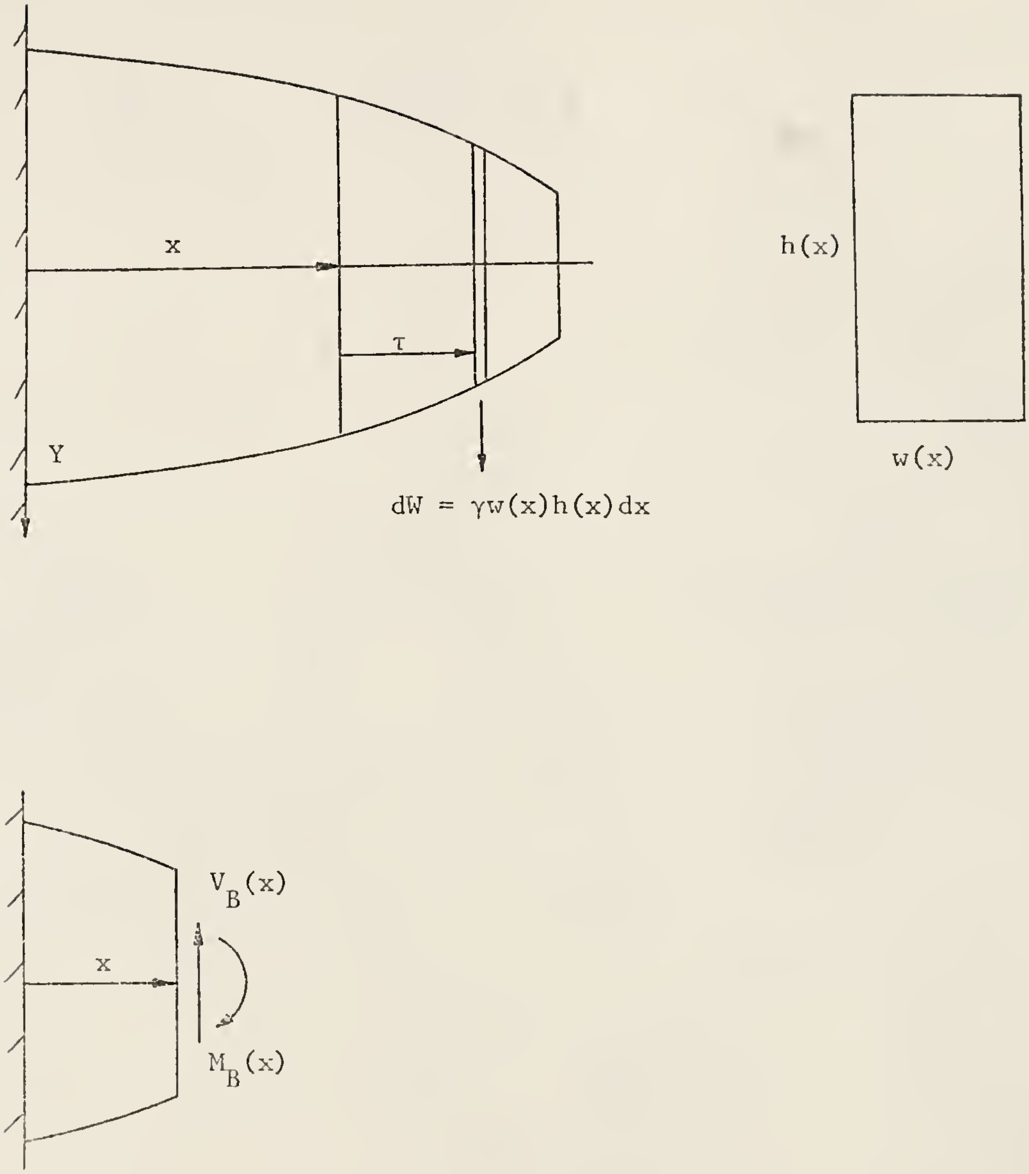
$$M_B(x) = \gamma \int_x^L (\tau - x) w(\tau)\ h(\tau) d\,\tau$$

$$I_B(x) = \frac{1}{12}\, w(x)\ h^3(x)$$

and $\gamma = \rho g$. Kinematic boundary conditions to be satisfied by the solution of (4.2.1) are:

$$Y(0) = 0$$

$$Y'(0) = 0$$



Note: Y(x) is centerline deflection, positive downward.

Figure 4.1 Structural Conventions

Design variables and the related constraints (4.1.1) are put into dimensionless form such that

$$v_1 = \frac{h(x)}{d} \quad \rightarrow \quad b/d \leq v_1 \leq 1$$

$$v_2 = \frac{w(x)}{c} \quad \rightarrow \quad a/c \leq v_2 \leq 1$$

and

$$I_B(x) = \frac{1}{12}\, cd^3\, v_1 v_2$$

Replacing the independent variable with a dimensionless equivalent, and using the control components allows the governing equation to be put into a dimensionless form. For

$$t = \frac{x}{L}$$

$$\frac{1}{12}\, \frac{E}{\gamma L^2} \left(\frac{d}{L}\right)^2 u_1^3 u_2 \ddot{Y} = \int_t^1 (\tau - t)\, u_1(\tau) u_2(\tau) d\tau \qquad (4.2.2)$$

where

$$u_i(t) = v_i(x(t))$$

When constant C_B is defined, the usual kinematical relationships for a beam may be written in a simple dimensionless form; that is, let

$$C_B = \frac{1}{12}\, \frac{E}{\gamma L^2} \left(\frac{d}{L}\right)^2 \qquad (\text{Units} = \text{Length}^{-1})$$

$$x_1 = C_B Y \qquad \text{Deflection}$$

$$x_2 = C_B \dot{Y} \qquad \text{Slope}$$

$$x_3 = C_B u_1^3 u_2 \ddot{Y} \qquad \text{Moment}$$

$$x_4 = C_B \frac{d}{dt} (u_1^3 u_2 \ddot{Y}) \qquad \text{Shear}$$

$$u_1 u_2 = C_B \frac{d^2}{dt^2} (u_1^3 u_2 \ddot{Y}) \qquad \text{Load}$$

These state component definitions are used with the natural boundary conditions to obtain

$$x_3(1) = 0$$
$$x_4(1) = 0$$

From (4.2.2), the state component definitions, and the above boundary conditions it follows that

$$\dot{x}_1 = x_2 \qquad x_1(0) = 0$$
$$\dot{x}_2 = x_3/u_2^3 u_2 \qquad x_2(0) = 0$$
$$\dot{x}_3 = x_4 \qquad x_3(1) = 0$$
$$\dot{x}_4 = u_1 u_2 \qquad x_4(1) = 0$$

These equations and boundary conditions are used in the following section to precisely state the problem. The solution and results are given later.

4.3 Unmodified Application of the Maximum Principle

In terms of the state variables defined in the preceding section, the problem can be stated with more mathematical precision. Find

$$\underline{u}_{OPT}(t) = \underset{\underline{u}\,\varepsilon\,\Omega}{\text{ARGMIN}}\,[x_1(1)]$$

subject to:

(i) differential constraints $\quad \underline{\dot{x}} = \underline{f}(\underline{x},\underline{u})$

(ii) kinematic boundary conditions $\quad x_1(0) = x_2(0) = 0$

natural boundary conditions $\quad x_3(1) = x_4(1) = 0$

(iii) hard geometric constraints $\quad b/d \le u_1(t) \le 1$

$\quad a/c \le u_2(t) \le 1$

According to terminology in the calculus of variations this is a Mayer type problem. Among the early papers concerning the PMP, Rozonoér (1959) applies the PMP to a similar problem giving a geometric interpretation to the function of the adjoint variables.

In Rozonoér's problem the cost is a generalization of the ordinary Mayer problem, in the sense that the cost function is a linear combination of the terminal state components. It can be shown via the calculus of variations that to minimize

$$J = \underline{c}_r^T \underline{x}(t_F)$$

where $\underline{c}_r$ is a vector of prescribed constants, the necessary conditions are

$$H \triangleq \underline{p}^T(t)\underline{f}(\underline{x},\underline{u},t)$$

$$\dot{\underline{x}} = H_{\underline{p}} = \underline{f}$$

$$\dot{\underline{p}} = -H_{\underline{x}} = -\underline{f}_{\underline{x}}^T \underline{p}$$

$$0 = H_{\underline{u}}$$

$$0 = \underline{p}^T(0)\ \underline{x}(0)$$

$$0 = \underline{p}^T(t_F)[\underline{c}_r + \underline{p}(t_F)]$$

$$H(\underline{x}_{OPT}, \underline{u}_{OPT}, \underline{p}, t) \geq H(\underline{x}_{OPT}, \underline{u}, \underline{p}, t)$$

That is,

$$\underset{\underline{u}}{\mathrm{Min}}[J] \rightarrow \underset{\underline{u}}{\mathrm{Max}}\ [H]$$

If we consider the cantilever beam problem, the forms in the necessary conditions are

$$\underline{c}_r = [1\ 0\ 0\ 0]^T$$

$$H = p_1x_2 + p_2x_3/u_1^3u_2 + p_3x_4 + p_4u_1u_2$$

$$t_F = 1$$

$$\underline{f}(\underline{x},\underline{u}) \begin{Bmatrix} x_2 \\ x_3/u_1^3u_2 \\ x_4 \\ u_1u_2 \end{Bmatrix}$$

from which

$$\dot{p}_1 = 0 \qquad p_1(1) = -1$$

$$\dot{p}_2 = -p_1 \qquad p_2(1) = 0$$

$$\dot{p}_3 = -p_2/u_1^3u_2 \qquad p_3(0) = 0$$

$$\dot{p}_4 = -p_3 \qquad p_4(0) = 0$$

Adjoint variables $p_1(t)$ and $p_2(t)$ can be integrated by inspection

$$\left.\begin{aligned} p_1(t) &= -1 \\ p_3(t) &= -\ (1-t) \end{aligned}\right\} 0 \le t \le 1 \qquad (4.3.1)$$

such that

$$H = -\ x_2 - (1-t)x_3/u_1^3u_2 + p_3x_4 + p_4u_1u_2$$

With this result, the necessary condition for control to minimize tip deflection is

$$H_{u1} = 3(1-t)x_3/u_1^4u_2 + p_4u_2 = 0$$

$$H_{u2} = (1-t)x_3/u_1^3u_2^2 + p_4u_1 = 0$$

At first this appears to be a contradiction since the two equations can be satisfied only by the trivial solution because they have equivalent forms,

$$\left.\begin{aligned} p_4u_1^4u_2^2 &= -3(1-t)x_3 \\ p_4u_1^4u_2^2 &= -\ (1-t)x_3 \end{aligned}\right\} \qquad (4.3.2)$$

Further examination, however, leads to the conclusion that when the control is completely unconstrained there is no horizontal tangent plane to the surface $H = H(u_1,u_2)$.

When the geometric constraints to the control are included, a constrained minimum may exist. If such is the case, the maximum value of H occurs on the boundary of admissible control space. To that end, PMP is employed along the control space boundary to determine $\underline{u}_{OPT}(t)$ at each time t. Before detailing this procedure, it is necessary to first consider some structural aspects of the problem.

By definition the control components are positive, which in turn implies

$$\text{Load:} \quad \underline{\dot{x}}_4 > 0 \leftarrow \dot{x}_4 = u_1u_2$$

$$\text{Shear:} \quad x_4 \leq 0 \leftarrow \dot{x}_4 > 0 \ , \quad x_4(1) = 0$$

$$\text{Moment:} \quad x_3 \geq 0 \leftarrow \dot{x}_3 < 0 \ , \quad x_3(1) = 0$$

Furthermore, since $p_2(t) \leq 0$ from (4.3.1)

$$p_3 > 0 \leftarrow \dot{p}_3 > 0 \ , \quad p_3(0) = 0$$

$$p_4 < 0 \leftarrow \dot{p}_4 < 0 \ , \quad p_4(0) = 0$$

This exercise makes it possible to use the information of the sense for x_3 and p_4 to simplify the search for $\underline{u}_{OPT}$ on the control boundary. By arranging the Hamiltonian in the following fashion

$$H = -x_2 + p_3x_4 + p_4[u_1u_2 - (1-t)x_3/p_4u_1^3u_2]$$

it is observed that both terms in the bracketed expression are positive. This and the p_4 outside the leading bracket allows the following equivalence:

$$\underset{\underline{u}\,\varepsilon\,\partial U}{\text{Max}}\ [H] \equiv \underset{\underline{u}\,\varepsilon\,\partial U}{\text{Min}}\ [u_1u_2 - (1-t)x_3/p_4u_1^3u_2]$$

or,

$$\underline{u}_{OPT} = \underset{\underline{u}\,\varepsilon\,\partial U}{\text{ARGMIN}}\ [\Phi(\underline{u})]$$

where

$$\Phi(\underline{u}) = [u_1u_2 + F^2(t)/u_1^3u_2\ ] \qquad (4.3.3)$$

and

$$F^2(t) = -\ (1-t)x_3/p_4 \geq 0$$

At each position t along the beam, the state and adjoint variables must satisfy the appropriate differential equations, and $\underline{u}$ is specified by the preceding three equations.

Control space boundary ∂U is illustrated in Figure 4.2, where the u_1 axis is treated as the ordinate since $u_1(t)$ and $u_2(t)$ correspond

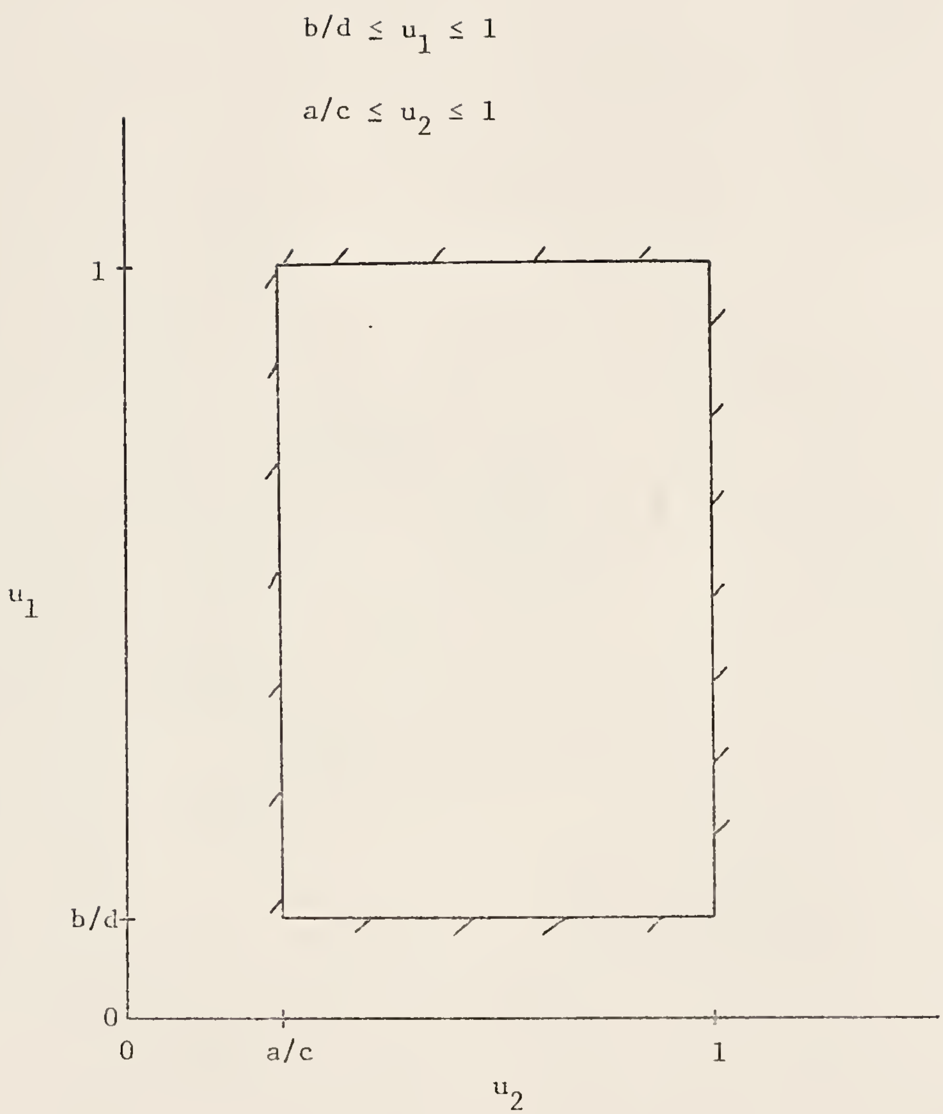


Figure 4.2 Admissible Control Space

to the height and width, respectively, of the cross section of the beam at position t.

Along the constant u_1 edges of ∂U, let $u_1 = u_c$ where u_c has the value of either b/d or unity. If $\Phi(u_c, u_2)$ has a minimum point

$$\frac{d\Phi}{du_2} = 0 \qquad \text{and} \qquad \frac{d^2\Phi}{du_2^2} > 0$$

where

$$\Phi(u_c, u_2) = u_c u_2 + F^2(t)/u_c^3 u_2$$

$$\frac{d\Phi}{du_2} = u_c \quad - F^2(t)/u_c^3 u_2^2$$

$$\frac{d^2\Phi}{du_2^2} = \qquad 2F^2(t)/u_c^3 u_2^3$$

The value of u_2 which satisfies the first condition is

$$u_2^* = F(t)/u_c^2$$

Furthermore, it is observed that only one extremum of $\Phi(\underline{u})$ exists along $u_1 = u_c$ and that it is a minimum. Hence, either $\Phi(u_c, u_2)$ has a minimum on the constant u_1 edge or is monotonically decreasing/increasing. If either

$$u_2^* \leq a/c \qquad \text{or} \qquad 1 \leq u_2^*$$

then along the constant u_1 edge, H has its maximum value at a corner of the rectangular ∂U. On the other hand, if

$$a/c < u_2^* < 1$$

then H has its maximum value on the line $u_1 = u_c$ interior to the endpoints.

Similarly, along the u_2 edges of ∂U, denote $u_2 = u_c$ where u_c has the value of either a/c or unity. If $\Phi(u_1,u_c)$ has a minimum point

$$\frac{d\Phi}{du_1} = 0 \qquad \text{and} \qquad \frac{d^2\Phi}{du_1^2} > 0$$

where

$$\Phi(u_1,u_c) = u_1u_c + F^2(t)/u_1^3u_c$$

$$\frac{d\Phi}{du_1} = \qquad u_c - 3F^2(t)/u_1^4u_c$$

$$\frac{d^2\Phi}{du_1^2} = \qquad 12F^2(t)/u_1^5u_c$$

It is observed that $\Phi(u_1,u_c)$ has only one extremum along $u_2 = u_c$, it is a minimum, and occurs at the point $u_1 = u_1^*$ where

$$u_1^* = + \{3F^2(t)/u_c^2\}^{\frac{1}{4}}$$

Thus, by the same argument posed in the preceding paragraph, if either

$$u_1^* \leq b/d \qquad \text{or} \qquad 1 \leq u_1^*$$

then along the constant u_2 edge, H has its maximum value at a corner of the rectangular ∂U. Wherever

$$b/d < u_1^* < 1$$

H has its maximum value on the line $u_2 = u_c$ interior to the endpoints.

On the basis of these arguments, the following system was solved by the method of quasilinearization:

$$\dot{x}_1 = x_2 \qquad x_1(0) = 0$$

$$\dot{x}_2 = x_3/u_1^3u_2 \qquad x_2(0) = 0$$

$$\dot{x}_3 = x_4 \qquad x_3(1) = 0$$

$$\dot{x}_4 = u_1u_2 \qquad x_4(1) = 0$$

$$\dot{p}_3 = -p_2/u_1^3u_2 \qquad p_3(0) = 0$$

$$\dot{p}_4 = -p_3 \qquad p_4(0) = 0$$

$$\underline{u}_{OPT} = \underset{\underline{u}\,\varepsilon\,\partial U}{\mathrm{ARGMIN}}\,[u_1u_2 - (1-t)x_3/p_4u_1^3u_2]$$

The beam is represented by 100 intervals composing the range $0 \leq t \leq 1$, which is separated by 101 "mesh points." An initial guess of the solution x(t) and p(t) is chosen; it is selected to satisfy the boundary conditions. This guess is not a solution and does not satisfy the differential equations. The $\dot{\underline{x}}$ and $\dot{\underline{p}}$ equations are linearized about the initial guess, then the resulting linear TPBVP is solved to obtain new $\underline{x}(t)$ and $\underline{p}(t)$ functions which more closely satisfy the differential equations. At each time t corresponding to a mesh point, H is numerically evaluated along each of the four straight line segments composing ∂U to determine $\underline{u}_{OPT}$. The point (u_1,u_2) on ∂U which gives $H(\underline{u};\underline{x},\underline{p},t)$ its maximum value is $\underline{u}_{OPT}$. This process is repeated until the $\underline{x}(t)$ and $\underline{p}(t)$ iterate satisfies the differential equations to within a specified tolerance. The equations necessary to use the IBM program available are given in Appendix C, in the form of a subroutine listing.

4.4 Results: Geometric Control Constraints

For the most part, no major difficulties were encountered in using quasilinearization to obtain a solution to the sixth order system derived in the previous section. Certain parameter values did engender numerical instability. These cases, the source of the difficulty, and its circumvention are discussed in Chapter VII. Moreover, all calculations were done in double precision as necessitated by matrix inversion accuracy requirements.

Parameter values selected to illustrate the solution method are:

$$\begin{aligned} u_1 &: \quad b/d = 0.25 \\ u_2 &: \quad a/c = 0.20 \end{aligned} \tag{4.4.1}$$

The measure of error of satisfaction of the differential equations in the TPBVP is in terms of the general system

$$\frac{dY_i}{dx} = f_i(\underline{Y}, x) \qquad Y \equiv \underline{Y}_i(x), \qquad i = 1,\ldots,n$$

$$\text{ERROR} = \underset{i}{\text{Max}} \left| dY_i - f_i(\underline{Y}, x)dx \right| \tag{4.4.2}$$

Deflection of a uniform beam due to its own weight was used to infer an initial guess which satisfies all boundary conditions:

$$\begin{aligned} x_1(t) &= t^4 \\ x_2(t) &= t^3 \\ x_3(t) &= 1 - t^2 \\ x_4(t) &= -1 + t \\ p_3(t) &= t^2 \\ p_4(t) &= -t^3 \end{aligned} \qquad 0 \leq t \leq 1 \tag{4.4.3}$$

With these specified parameter values and initial guess of the solution, the program converged to a solution in five iterations. From this run a tolerance was selected for all subsequent cases; the following tabulation provides the data used in its selection:

Iteration	ERROR	Tip Deflection (Cost)
1	.2028	$.7387749327 \times 10^{-1}$
2	.1704	$.3192152426 \times 10^{-1}$
3	$.6533 \times 10^{-1}$	$.2847993812 \times 10^{-1}$
4	$.3031 \times 10^{-5}$	$.2853731846 \times 10^{-1}$
5	$.1129 \times 10^{-10}$	$.2853719983 \times 10^{-1}$

It is seen from these tabular data that there is little improvement in cost (tip deflection) as a result of the fifth iteration. For this reason a value for the tolerance was selected as 0.5×10^{-6} which corresponds to about six significant digits in the cost functional.

Recall from the previous section that no unconstrained minimum exists. With the control bounds included, the intuitive solution is one in which the cross-sectional area is maximum near the root, and reduces to a minimum at the tip. Recalling that for the optimal control,

$$\underline{u}_{OPT} = \underset{\underline{u}\,\varepsilon\,\Omega}{\text{Max}}\ [H] \equiv \underset{\underline{u}\,\varepsilon\,\Omega}{\text{Min}}\ [\Phi]$$

A sequence of illustrations in Figure 4.3 demonstrates the location of $\underline{u}_{OPT}$ on ∂U for several stations along the beam. Constant contours of $\Phi(\underline{u})$ are plotted on the admissible control space at five distinct positions. If an extremal point exists interior to ∂U some lines of constant $\Phi(\underline{u})$ contours must be closed curves in $\underline{u}$-space. This is impossible for this example.

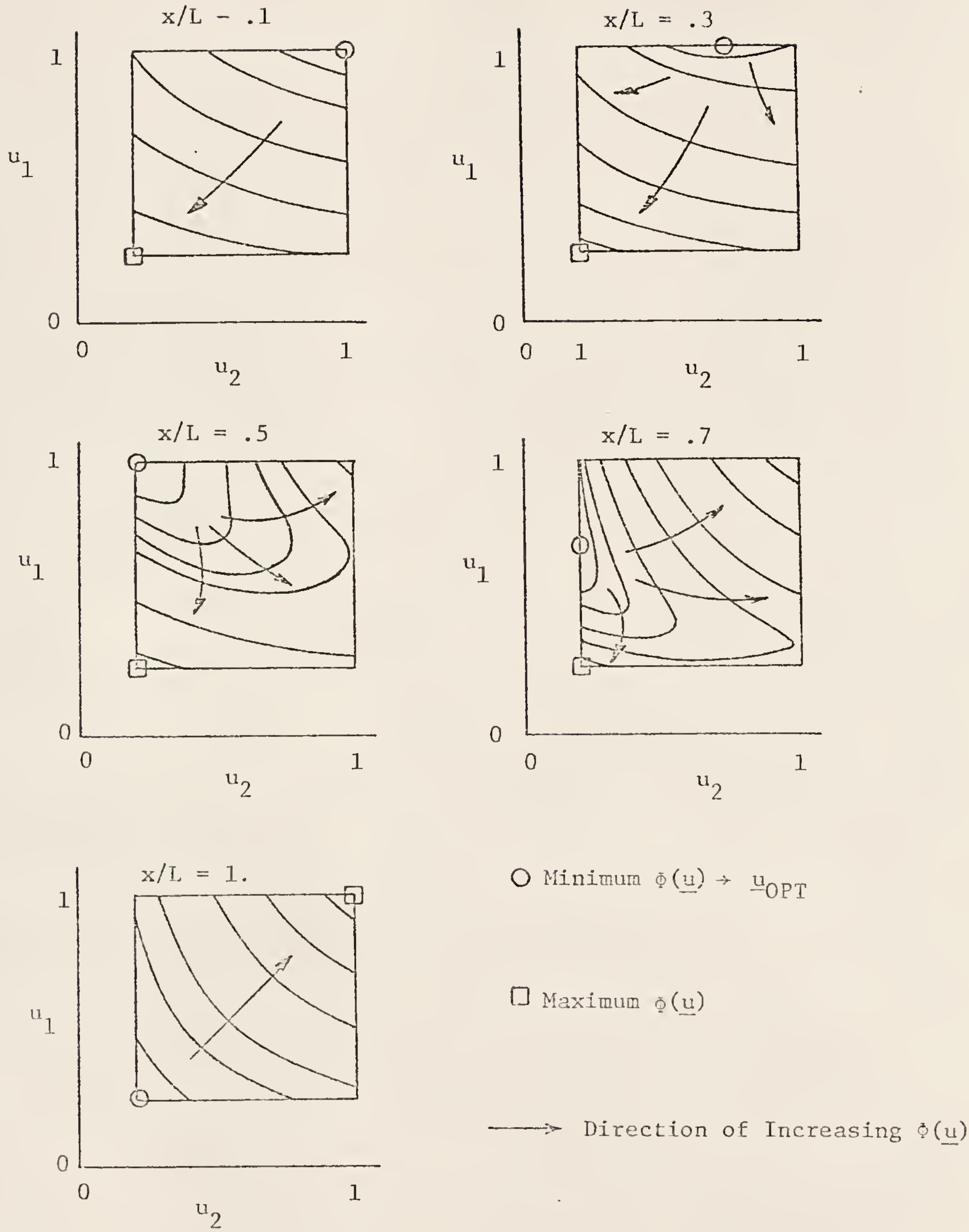


Figure 4.3 Contour Plots of $\Phi(\underline{u})$ at Various Stations Along the Beam

The first illustration is for the $t = 0.1$ cross section, near the root of the beam. Since $\underline{u}_{OPT}$ occurs at the point of minimum $\Phi(\underline{u})$, the optimum value for both u_1 and u_2 is unity, the maximum allowable dimensions for both height and width. Constant contour lines indicate that $\Phi(\underline{u})$ is mathematically decreasing in either direction of $\underline{u}$-space. Lines of constant $\Phi(\underline{u})$ are also plotted for the cross section of $t = 0.3$. The optimum control has the maximum admissible value for height u_1 but u_2 has a value somewhat less than unity. However, there are still no contour lines which are closed curves.

At the midpoint cross section the minimum $\Phi(\underline{u})$ point occurs at $u_1 = 1$ and $u_2 = a/c$. Although the surface $\Phi(\underline{u})$ forms a scoop-like shape, there are still no closed curve contours, and hence no extremal interior to admissible $\underline{u}$-space. The next cross section at which $\Phi(\underline{u})$ is displayed occurs at $t = 0.7$. On this section, u_2 is still at its lower bound but u_1 is no longer at the maximum allowable value of unity as shown in Figure 4.3. In the last of the sequence, $\Phi(\underline{u})$ contours for the cross section at the tip of the beam are displayed. The point of minimum $\Phi(\underline{u})$ occurs where both components of control have their minimum allowable values. Again, no contour lines of constant $\Phi(\underline{u})$ form a closed curve indicating the existence of an interior extremal point.

This sequence of illustrations indicates two things. First, the lack of closed curve contour lines of $\Phi(\underline{u})$ verifies that $\underline{u}_{OPT}$ exists on ∂U. With further study it may be possible to obtain some condition on $H_{\underline{u}} = 0$ which implies the equations corresponding to (4.3.2) can never yield a finite, unconstrained optimum. Such a condition would define the class of structures whose unconstrained solution is the

"zero volume solution" frequently described in the literature on structural optimization. Secondly, at $t = 0$ the point $\underline{u}_{OPT}$ occurs at $\underline{u} = [1,1]^T$, the point of maximum cross-sectional area; as t increases from zero to one, the point $\underline{u}_{OPT}$ moves along the $u_1 = 1$ boundary of ∂U to the u_2 lower bound, and then down the $u_2 = a/c$ boundary of ∂U to u_1 lower bound. By the time $t = 1$ the optimal cross-sectional area is the minimum allowable area. As a result of the prescribed form of ∂U, if $\underline{u}_{OPT}(t)$ follows this particular path as t increases from zero to one, each component of $\underline{u}_{OPT}(t)$ has its own distinct region of transition. That is, at any value of t, if $b/d < u_1 < 1$, then u_2 must be on either its upper or lower bound. Conversely, if u_2 is in transition where $a/c < u_2 < 1$, then u_1 must be on one of its bounds.

This effect is seen most clearly in Figure 4.4, where $\underline{u}_{OPT}$ is displayed for the example case parameter values specified by (4.4.1). The profiles are displayed on a two-view drawing as a plan-form of the beam might appear. State components corresponding to this beam are shown in Figure 4.5, representing dimensionless deflection, slope, moment, and shear, respectively. As observed in Figure 4.4, there are five distinct regions of the beam:

(i) $0 \leq t \leq .25$

$u_1 = 1$

$u_2 = 1$ controls onupper bounds

(ii) $.25 < t < .52$

$u_1 = 1$

$a/c < u_2 < 1$ u_2 transition

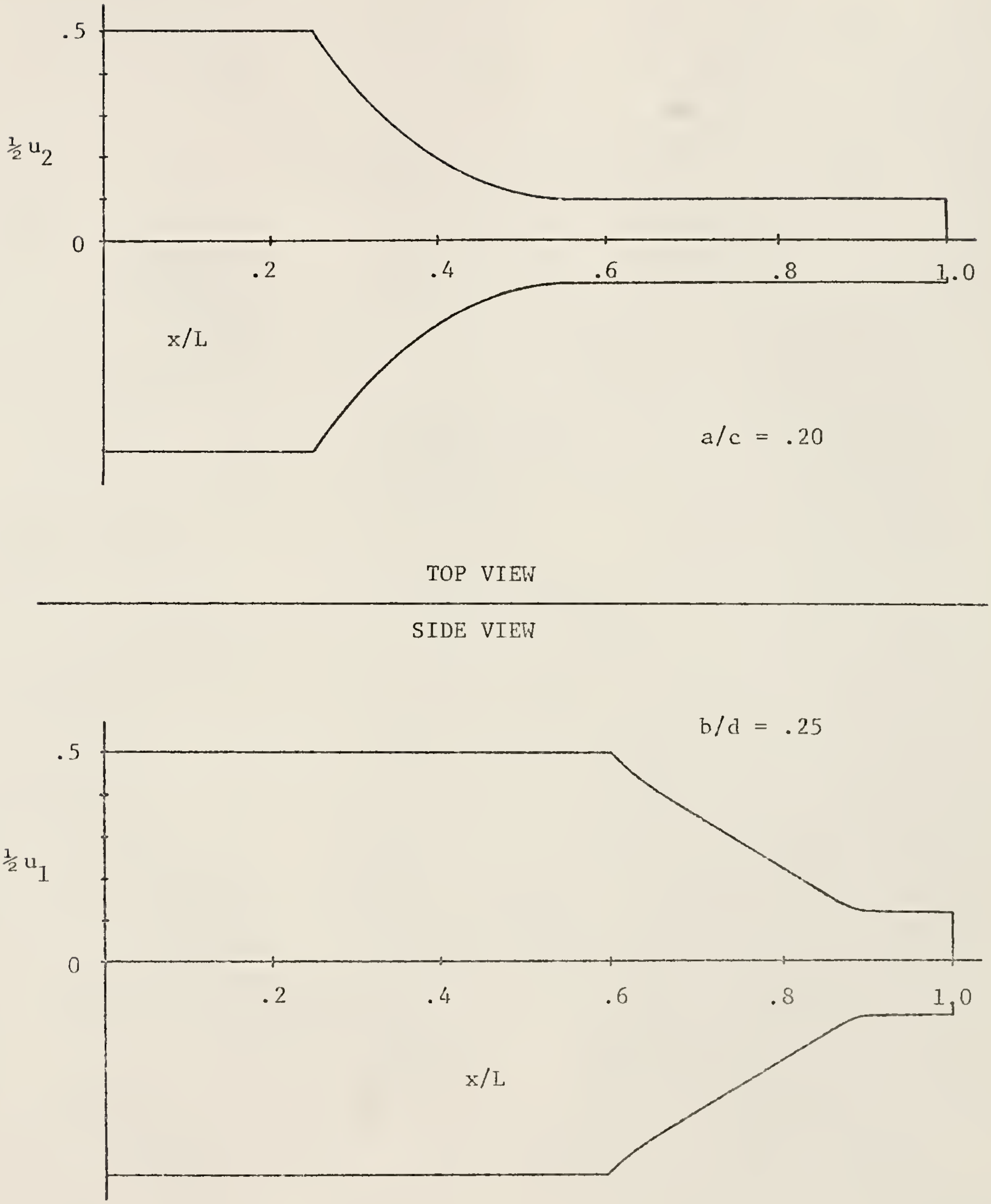


Figure 4.4 Plan-form Views of Optimal Solution for $b/d = .20$ and $a/c = .25$

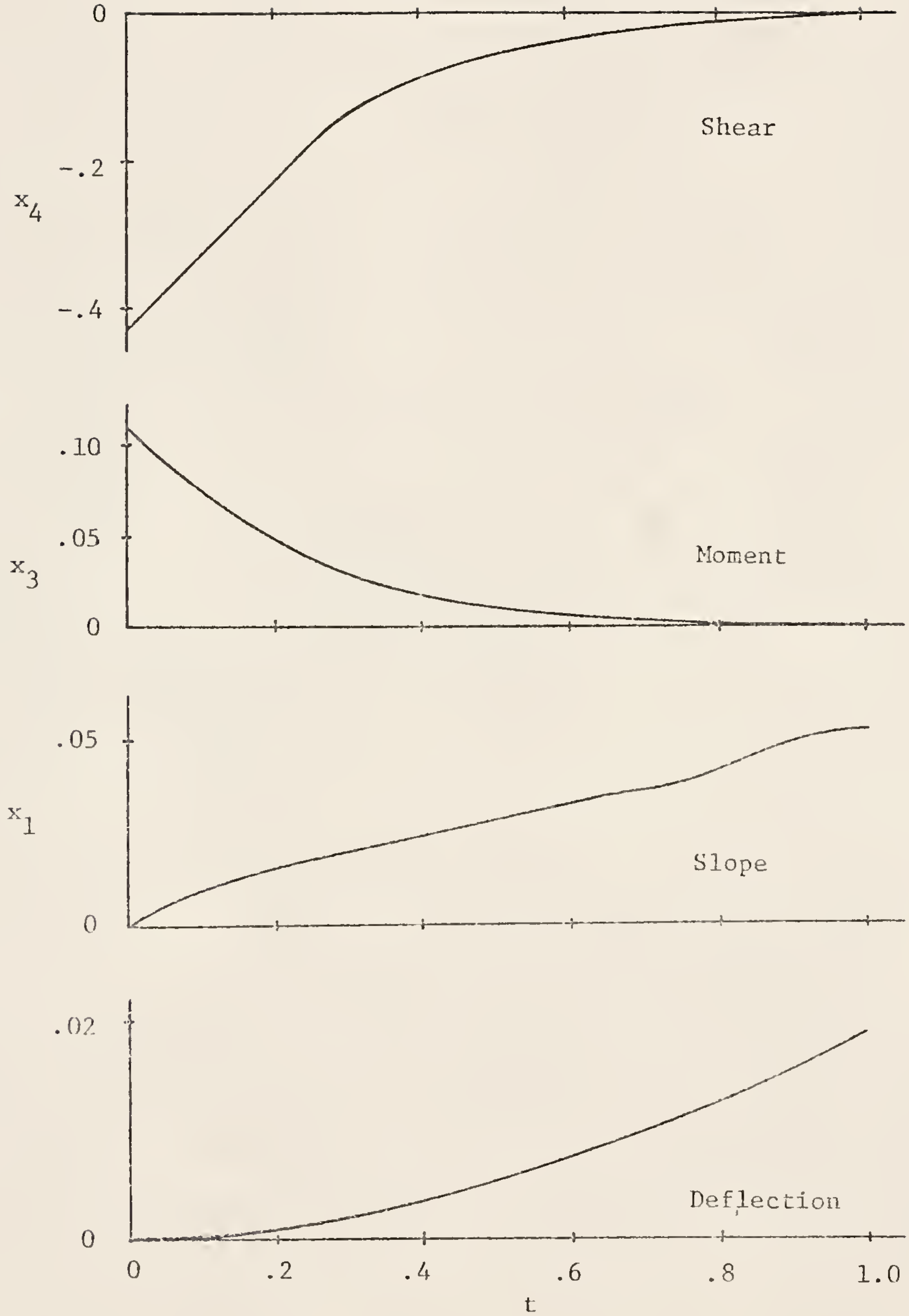


Figure 4.5 State Components of Optimal Solution for b/d = .20 and a/c = .25

(iii) $.52 \leq t \leq .59$

$u_1 = 1$ on upper bound

$u_2 = a/c$ on lower bound

(iv) $.59 < t < .90$

$b/d < u_1 < 1$

$u_2 = a/c$

u_1 transition

(v) $.90 \leq t \leq 1.00$

$u_1 = b/d$

$u_2 = a/c$

controls on lower bounds

The curves that show the intercept locations as a function of parameter values b/d have been called "correlation curves" in earlier studies. When the width is allowed to vary also, the second parameter a/c is introduced. For the sake of comparison to previous studies, the intercept/correlation curves are plotted as dependent upon b/d and parametric in a/c. However, it would be just as correct to do the opposite.

Intercept location curves described above are shown in Figure 4.6. The heavy black curve is the case where $a/c = 1$, a beam of constant uniform width--the case cited from earlier literature. Another special case is represented by dashed lines, corresponding to $a/c = 0$ which is the case corresponding to a minimum allowable thickness equal to zero. Dashed lines are used because these data are an extrapolation: convergence problems encountered for parameter values less than 0.1 prevented obtaining numerical results.

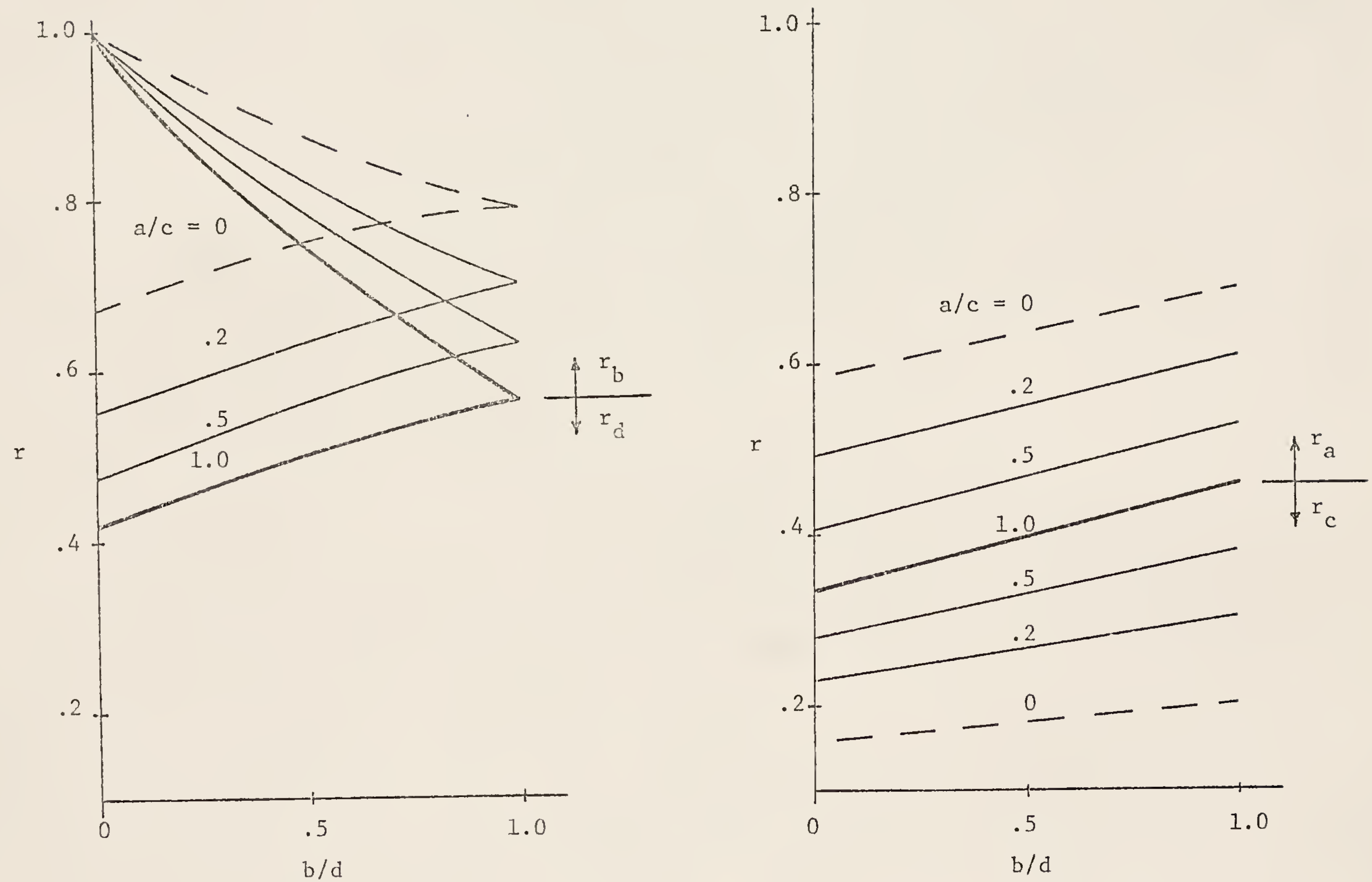


Figure 4.6 Intercept Location (Correlation Curves)

A discussion on the convergence difficulties experienced by the quasilinearization algorithm for parameter values approaching zero is presented in the chapter on numerical instabilities. In that discussion, isolation of the source of difficulty is reported; it is possible that this difficulty may be a general result applicable to all problems to be solved by the method of quasilinearization. A solution for this case is later obtained by finite element techniques. Note that since $0 \leq a/c \leq 1$ these two cases represent limits to the solutions of the problem. In addition, if the four intercept locations are plotted versus a/c and parametric in b/d, curves r_a and r_c appear as "horizontal vees" with r_b and r_d lines that are nearly parallel.

It is interesting to note from the figure depicting the solution of this case as a plan-form, that in the central region of the beam, the height is greater than the width. This result can be anticipated since such a configuration gives a greater bending resistance per unit weight.

With further reference to Figure 4.4, the transition of $u_1(t)$ is seen to be almost a linear taper, whereas the $u_2(t)$ transition exhibits a much more pronounced curvature. To generalize from this specific case of given values of b/d and a/c to arbitrary values requires the introduction of four quantities characterizing the solution. These quantities are the values of t at the points where the transitions intercept the bounds on u_1 and u_2; since t represents a normalized position x/L, these quantities can be thought of as an intercept location expressed as percent of the beam's length. They

are defined with reference to the five distinct regions of the beam previously given, where

r_b designates $u_1(t)$ intercept with lower bound

r_d designates $u_1(t)$ intercept with upper bound

r_a designates $u_2(t)$ intercept with lower bound

r_c designates $u_2(t)$ intercept with upper bound

such that the five regions are:

(i)	$0 \leq t \leq r_c$	control on upper bounds
(ii)	$r_c < t < r_a$	u_2 transition
(iii)	$r_a \leq t \leq r_d$	control on upper/lower bounds
(iv)	$r_d < t < r_b$	u_1 transition
(v)	$r_b \leq t \leq 1.0$	control on lower bounds

4.5 Inequality Stress Constraints

This section treats an inequality limit to allowable normal and shear stresses associated with bending. Using the ordinary strength of materials formulations it can be shown that for the rectangular cross section these constraints take the form

$$\frac{1}{2} \frac{M_B(x)}{I_B(x)} h(x) \leq \sigma_{MAX}$$

and

$$-\frac{1}{8} \frac{V_B(x)}{I_B(x)} h^2(x) \leq \tau_{MAX}$$

When dimensionless quantities are introduced

$$6 \frac{\gamma L^2}{d} x_3/u_1^2 u_2 \leq \sigma_{MAX}$$

$$-\frac{3}{2} \gamma L \; x_4/u_1 u_2 \leq \tau_{MAX}$$

the inequalities may be written in the required form for mixed constraints, i.e., as a function of both control and state components:

$$\phi_1(\underline{x},\underline{u}) = x_3/u_1^2u_2 - \sigma_0 \leq 0 \tag{4.5.1}$$

$$\phi_2(\underline{x},\underline{u}) = - x_4/u_1u_2 - \tau_0 \leq 0 \tag{4.5.2}$$

where

$$\sigma_0 = \frac{1}{6}\,\frac{\sigma_{MAX}}{\gamma L}\,(\frac{d}{L})$$

$$\tau_0 = \frac{2}{3}\,\frac{\tau_{MAX}}{\gamma L}$$

The two stress constraints place restrictions upon the minimum cross-sectional dimensions to keep the normal and shear stresses less than prescribed values. Specifically, from the constraints (4.5.1) and (4.5.2), two control inequalities must be satisfied at each station t, and these inequalities depend upon the state of the structural system. The inequalities are:

$$u_1^2u_2 \geq x_3/\sigma_0$$

$$u_1u_2 \geq - x_4/\tau_0 \quad , \qquad x_4(t) \leq 0$$

from which can be derived boundary arcs in $\underline{u}$-space:

$$\begin{aligned} u_{1\sigma}(u_2) &= (x_3/\sigma_0u_2)^{\frac{1}{2}} \geq 0 \\ u_{1\tau}(u_2) &= - x_4/\tau_0u_2 \geq 0 \end{aligned} \tag{4.5.3}$$

Both of these boundary arcs are hyperbolas restricted to the first quadrant of $\underline{u}$-space. Depending upon the location of the arcs, vis-à-vis the rectangular ∂U, inclusion of stress constraints has one of three

effects in determining what $\underline{u}$ is admissible. At any station t in some structural state x(t),

(i) if σ_0, τ_0 is too small the stress boundary arc lies entirely above rectangular ∂U; all geometrically admissible $\underline{u}$ violate the stress constraints.

(ii) if σ_0, τ_0 is too large the stress boundary arc lies entirely below rectangular ∂U; all geometrically admissible $\underline{u}$ satisfy the stress constraints.

(iii) for some range of σ_0, τ_0 the stress boundary arc divides the rectangular ∂U into two regions: the upper region consists of geometrically admissible $\underline{u}$ that satisfy the stress constraint, $\underline{u}$ in the lower region are geometrically admissible but violate the stress constraint.

Inclusion of stress constraints alters the admissible control space from the rectangular shape previously considered to a shape that may contain a stress boundary arc as part of its boundary. Consider the normal stress boundary arc specified by (4.5.3) to be a part of ΦU. Then to find $\underline{u}_{OPT}$ in the manner outlined in Section 4.3, $\Phi(\underline{u})$ must be evaluated along $u_{1\sigma} = u_{1\sigma}(u_2)$. If a minimum exists along the orthogonal projection of $u_{1\sigma}(u_2)$ on the $\Phi(\underline{u})$ surface, then

$$\underset{u_1}{\text{Min}} \left[\Phi(\underline{u}) \Big|_{u_{1\sigma}} \right] \rightarrow \begin{cases} \dfrac{\partial \Phi(\underline{u})}{\partial u_1} = 0 \\[2ex] \dfrac{\partial^2 \Phi(\underline{u})}{\partial u_1^2} > 0 \end{cases}$$

Along the normal stress boundary arc $u_{1\sigma}(u_2)$,

$$u_1^2 u_2 = x_3/\sigma_0 \rightarrow u_2 = x_3/\sigma_0 u_1^2$$

Substituting for u_2 in $\Phi(\underline{u})$ it follows from (4.3.3) that

$$\Phi(\underline{u})\Big|_{u_{1\sigma}} = (x_3/\sigma_0 + F^2(t)\sigma_0)/u_1$$

Since the expression in parentheses is positive, it is readily observed that the value $u_1 = +\infty$ minimizes $\Phi(\underline{u})$ along $u_{1\sigma}(u_2)$. From (4.5.3), the value of u_2 at this point in $\underline{u}$-space is zero.

The same result exists for $\Phi(\underline{u})$ evaluated along $u_{1\tau}$; on the shear stress boundary arc

$$u_1 u_2 = - x_4/\tau_0 \rightarrow u_2 = - x_4/\tau_0 u_1$$

Substituting for u_2 in $(\underline{u})$ it follows from (4.3.3) that

$$\Phi(\underline{u})\Big|_{u_{1\tau}} = (-x_4/\tau_0) + (-x_4/\tau_0)^{-1} F^2(t)/u_1^2$$

Since both the expression in parentheses and $F^2(t)$ are positive, it can be argued as above that the point of minimum $\Phi(\underline{u})$ along $u_{1\tau}(u_2)$ occurs at the point $(u_{1\tau} = +\infty,\ u_2 = 0)$.

At any cross section there exists a single minimum of $\Phi(\underline{u})$ along either stress constraint boundary arc. Since these minima occur at the point $\underline{u}$: $(+\infty, 0)$, as one proceeds along either boundary in the direction of increasing u_1, $\Phi(\underline{u})$ is a monotonically decreasing function. For this reason, along the stress constraint boundary arc, it is necessary to evaluate $\Phi(\underline{u})$ at only one point. That point is the intersection of the stress boundary constraint arc with the rectangular boundary having the

larger value of u_1. It is possible that the two stress boundaries intersect, with the coordinates of that point given by

$$u_{1I} = \left(\frac{x_3}{\sigma_0}\right)\left(\frac{\tau_0}{-x_4}\right)$$

$$u_{2I} = \left(\frac{\sigma_0}{x_3}\right)\left(\frac{-x_4}{\tau_0}\right)^2$$

Even though this adds a complication to the boundary of what is admissible $\underline{u}$, the monotonically decreasing property of both stress boundary arcs still results in the necessity of evaluating $\Phi(\underline{u})$ at only a single point along the stress boundary arcs. To illustrate this, Figure 4.7 depicts an admissible control region determined by a combination of geometric constraints and a composite stress constraint boundary, together with the point at which $\Phi(\underline{u})$ is a minimum along the composite stress boundary. With continued reference to the figure, $\underline{u}_{OPT}$ is that point along line ABC which minimizes $\Phi(\underline{u})$.

4.6 Results: Stress Constraints Included

For the purpose of comparison, the parameter values used in the case with only geometric constraints were also used in the cases where stress inequality constraints are present. The results are quite simply stated: neither stress constraint is ever active. In general it was found that the critical section occurs at the root. At that section the greatest portion of geometrically admissible control space is prohibited by the stress constraints. The further along the beam towards the tip, the less the prohibited geometrically admissible control space.

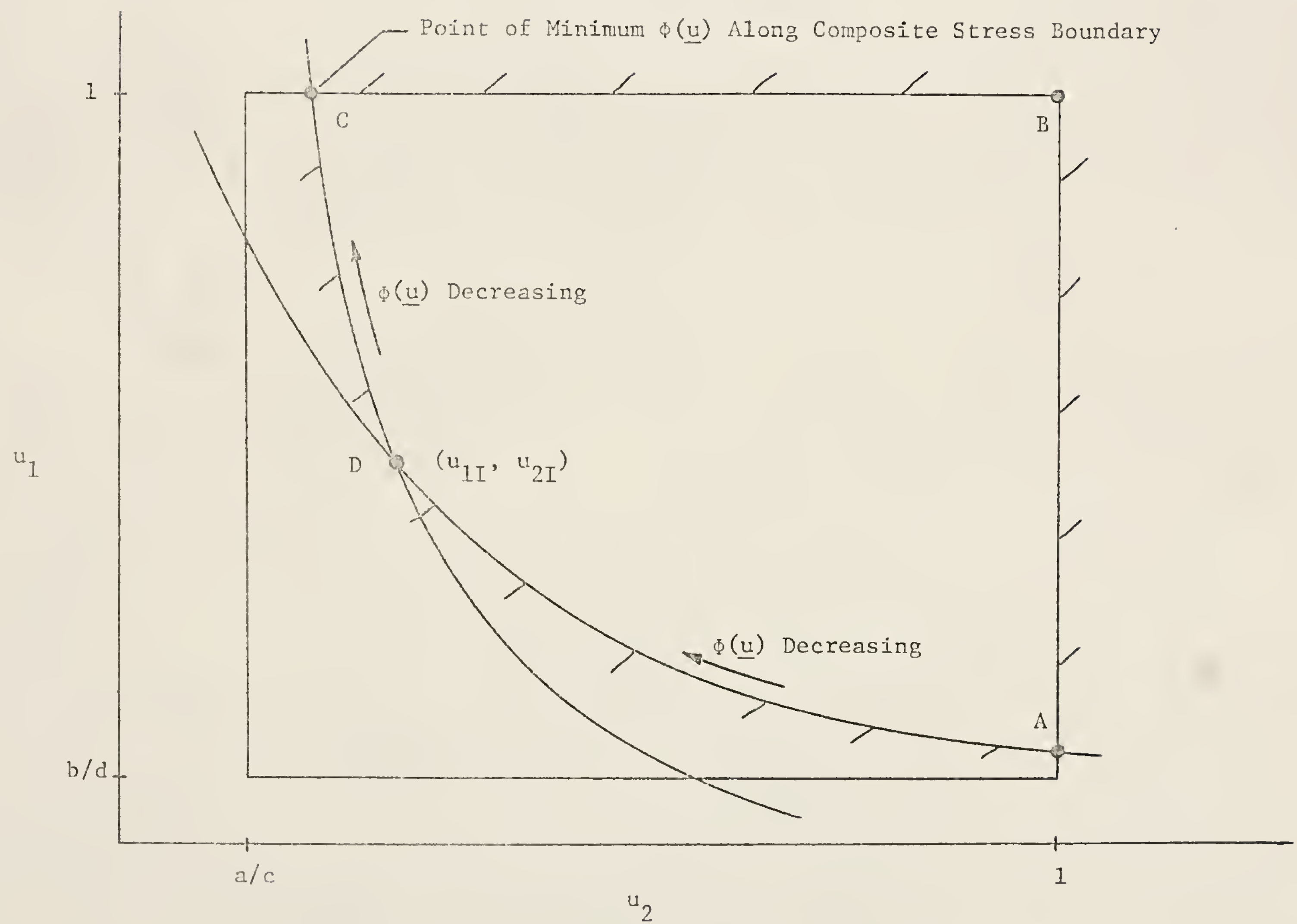


Figure 4.7 Behavior of $\Phi(\underline{u})$ Along Composite Stress Boundary

Both x_3 and $-x_4$ decrease monotonically to zero at $t = 0$, and is the reason for this phenomenon.

Each of the parameters σ_0 and τ_0 thus has three important critical values. For the first pair of values the maximum allowable stresses are sufficiently large that all geometrically admissible control contained by the rectangular ∂U satisfy the stress constraints. In this case

$$\sigma_0 = 9.128$$

$$\tau_0 = 8.658$$

and the stress constraints intersect the lower left-hand corner of ∂U at $t = 0$ and lie below ∂U for all $t > 0$. For the range of values

$$.5705 \leq \sigma_0 \leq 9.128$$

$$2.1645 \leq \tau_0 \leq 8.658$$

the stress boundary arcs intercept the $u_2 = .2$ portion of rectangular ∂U. If they are exactly equal to the lower values, respectively, both arcs pass through the point (1.,.2) at $t = 0$, and lie below that point for all $t > 0$. However, these values are not as important as either the first pair or the next. For the range of values

$$.1141 \leq \sigma_0 \leq .5705$$

$$.4329 \leq \tau_0 \leq 2.1645$$

both arcs intercept the $u_1 = 1$ portion of rectangular ∂U. When equal to the lower values both arcs pass through the upper right-hand corner of ∂U at $t = 0$. If the values of

$$\sigma_0 < .1141$$

$$\tau_0 < .4329$$

then at $t = 0$ there is no control which satisfies both geometric constraints and stress constraints. The problem has inconsistent constraints such that no solution is possible.

A sequence of plots showing the admissible control space is shown in Figure 4.8 for six separate stations along the beam. The value of σ_0 is .1141 for this case such that in section $t = 0$ the set of admissible controls is the single point $\underline{u}$: (1, 1). That portion of the geometrically admissible control space disallowed by the stress constraint is indicated by dashed lines. Additionally, the point corresponding to $\underline{u}_{OPT}$ is included for each section. By observing the sequence of plots as t increases, both the stress boundary and $\underline{u}_{OPT}$ are seen to move closer to the origin of $\underline{u}$-space. However, $\underline{u}_{OPT}$ never overtakes the stress boundary, thus the normal stress boundary never becomes active. Figure 4.9 depicts the same case for the shearing stress constraint.

Two additional figures are included with the intermediate critical values of σ_0 and τ_0. The normal stress constraint case $\sigma_0 = 0.5705$ is shown in Figure 4.10; shear stress constraint case $\tau_0 = 2.1645$, in Figure 4.11. No illustrations are provided for the cases associated with the largest critical values since for $t > 0$ all geometrically admissible controls satisfy the stress constraints.

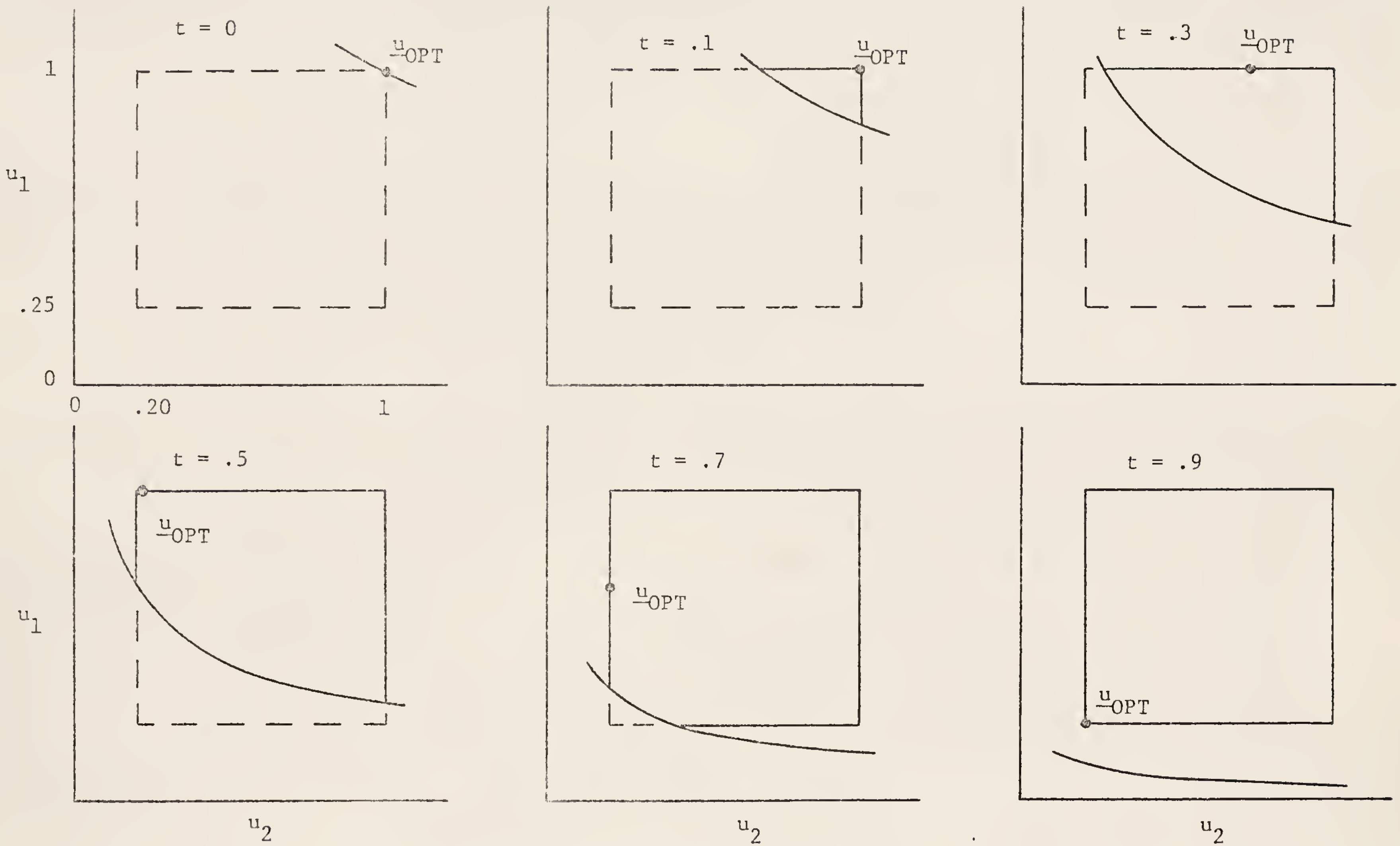


Figure 4.8 Admissible Control: $\sigma_o = .1141$

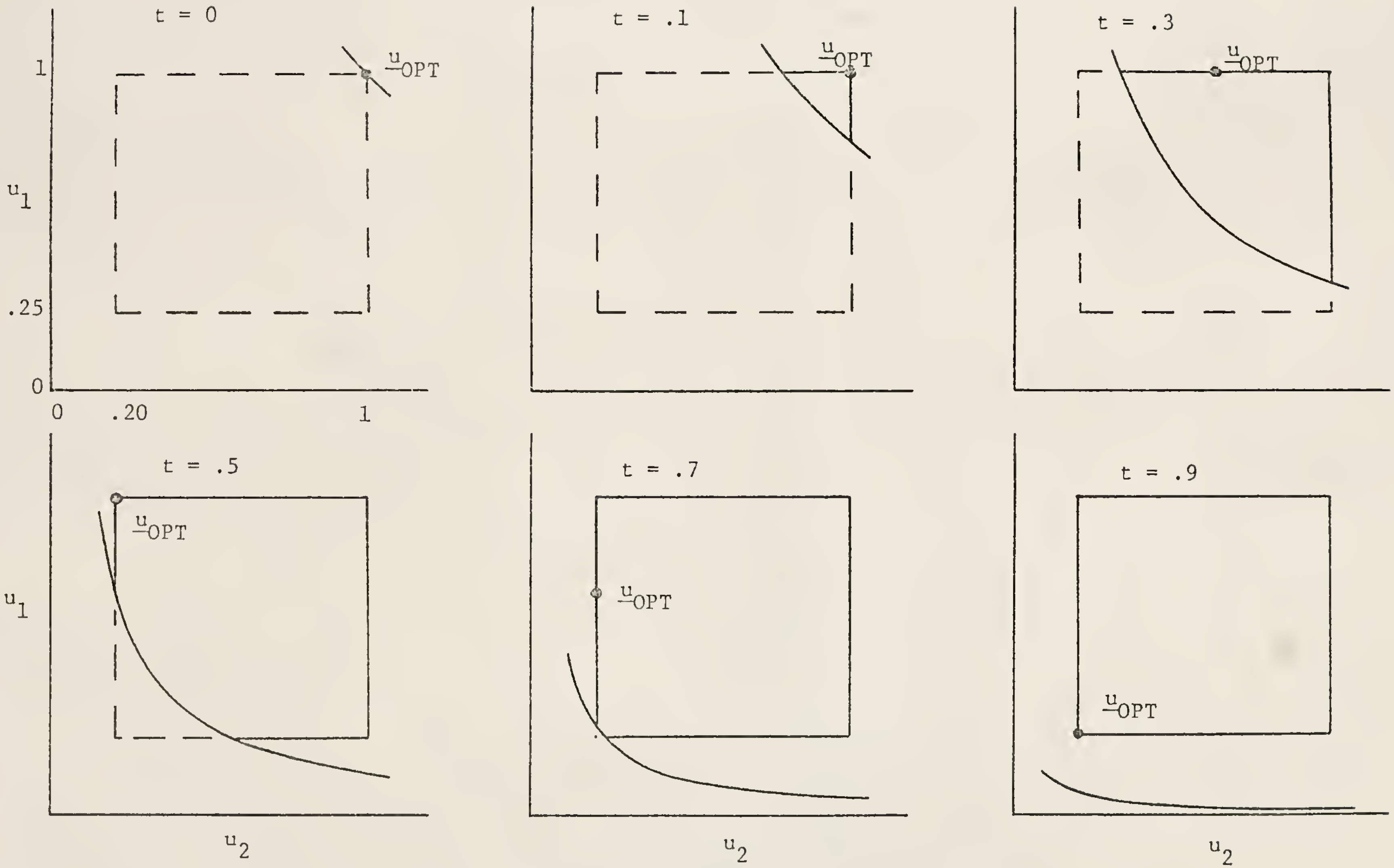


Figure 4.9 Admissible Control: $\tau_o = .4329$

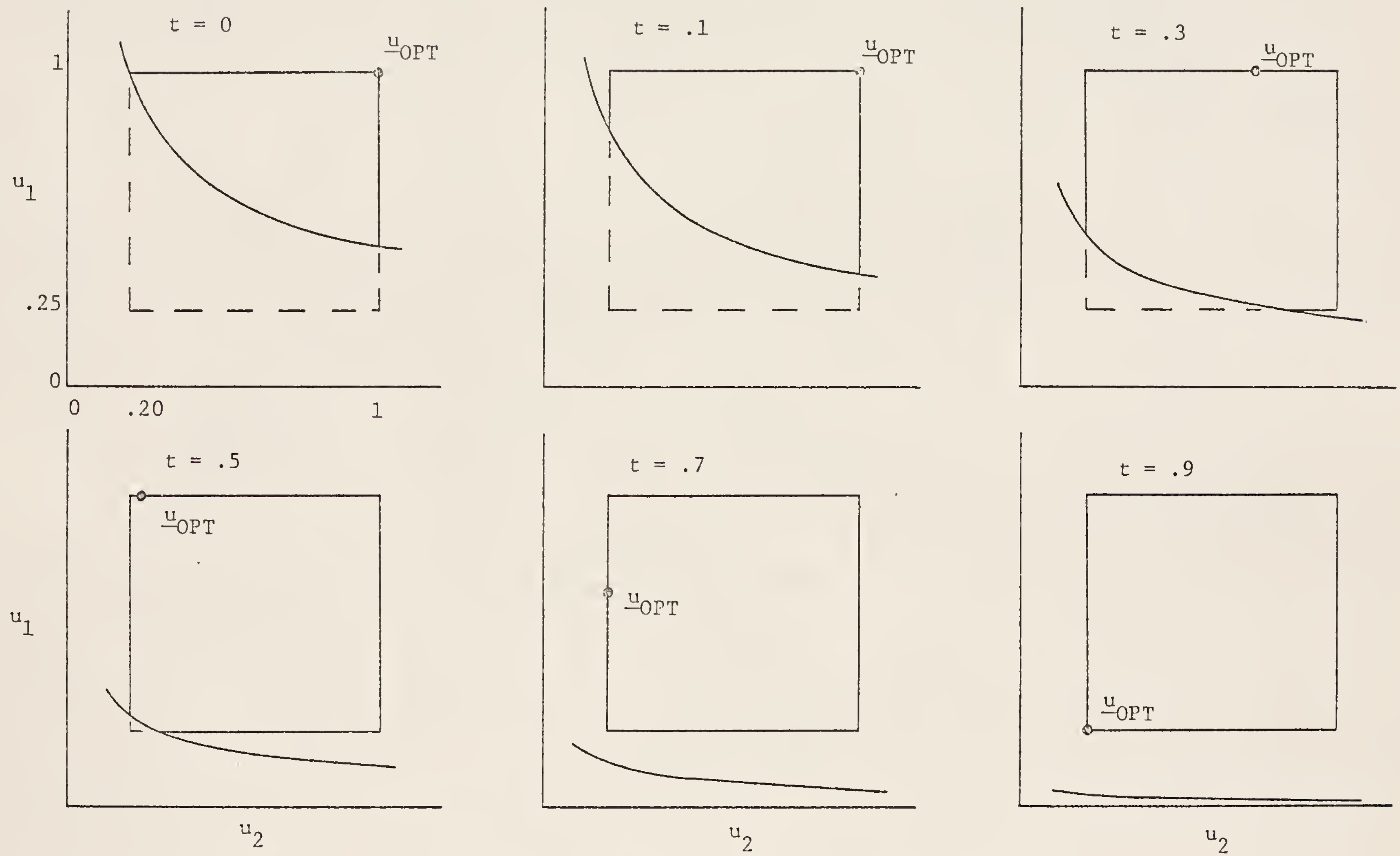


Figure 4.10 Admissible Control: $\sigma_o = .5705$

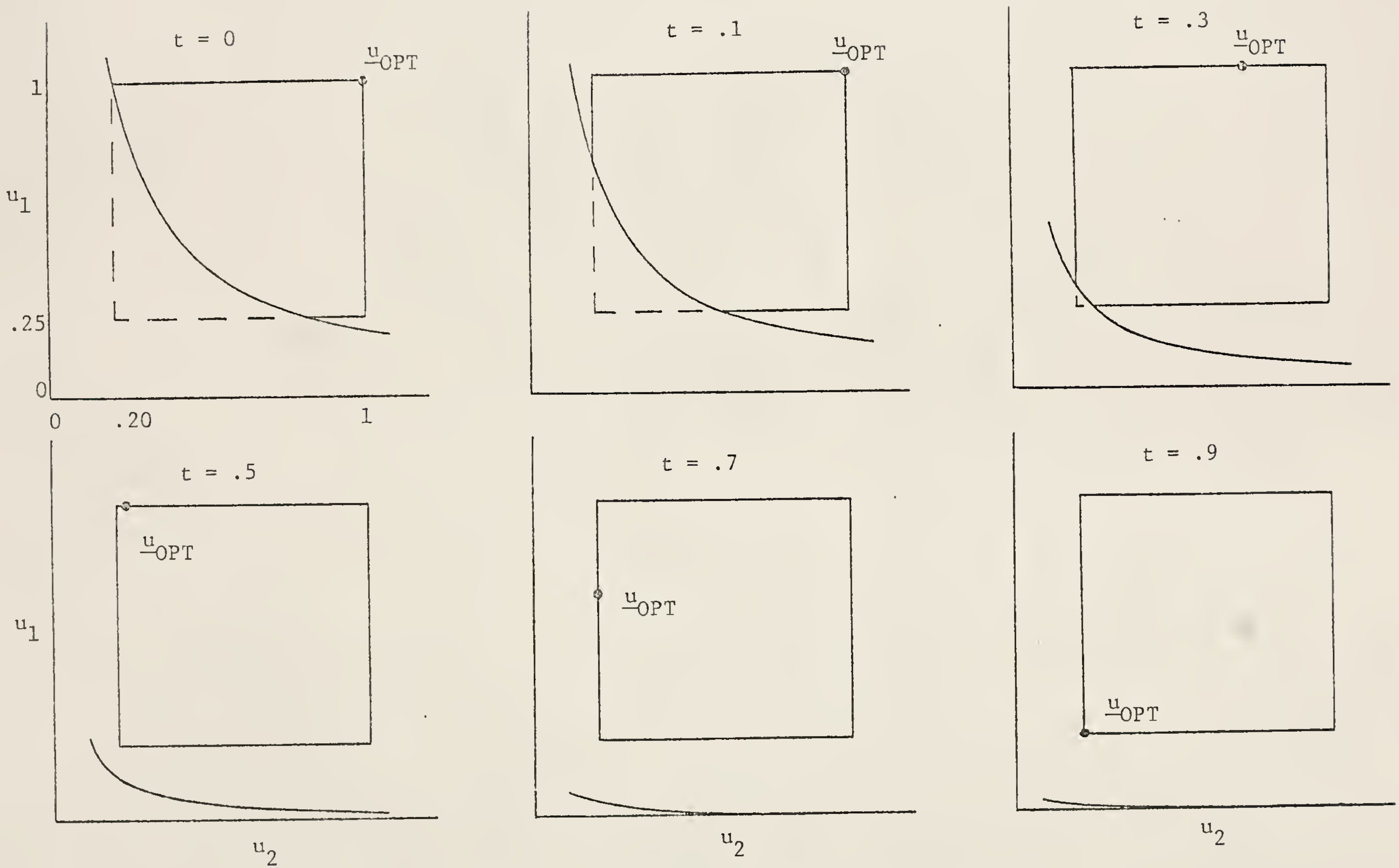


Figure 4.11 Admissible Control: $\tau_o = 2.1645$

CHAPTER V

CONSTRAINED DESIGN FOR AN OPTIMAL EIGENVALUE PROBLEM

5.0 Introduction

A column is to be designed for maximum buckling load in order to demonstrate (i) an application of the theory developed in Chapter III, and (ii) an optimal eigenvalue problem. Difficulties pertinent to this class of problems are discussed as well as a method to obtain a solution by using a theorem concerning self-adjoint problems. Use of this theorem with the maximum principle overcomes the difficulty associated with the simultaneous dual optimization required for problems where the cost functional is an eigenvalue expressed as a Rayleigh quotient. Numerical solutions to the combined set of state and adjoint variables are obtained by the method of quasilinearization. Characteristics of the solutions for a variety of control bounds are presented, along with a discussion of attempts to also include a mixed, inequality constraint related to stress.

5.1 Problem Statement

A vertical column, fixed at the base and free at the tip, is to be designed such that the buckling load is maximized. Length L is a specified constant as is the total weight W. The dependence of cross-sectional area A(x), material modulus E(x), and density $\rho(x)$, upon

position x along the column is to be such that the vertical end load P at buckling is maximized. Weight is not to be neglected and all cross-sectional shapes are similar.

In more specific terms, a given weight of material is to be arranged with variable properties and geometry such that the resulting mass distribution M(x) and stiffness distribution S(x) maximizes the vertical tip load the column can support when the effect of weight is included. Upper and lower bounds to all design variables

$$A_L \leq A(x) \leq A_U$$

$$E_L \leq E(x) \leq E_U \tag{5.1.1}$$

$$R_L \leq \rho(x) \leq R_U$$

are also specified. Governing equations for the structural system are derived in the next section.

5.2 Structural System

In the subsequent analysis, small deflection bending theory is assumed: the result is a linear differential equation with variable coefficients. Without this assumption, the governing equation is that of the nonlinear elastica problem. Additionally, axial compression effects are neglected as second order. Let Y(x) denote the deflection of the centerline at position x along the beam as depicted in Figure 5.1. Since the system is conservative, the governing equations can be derived quite simply by energy techniques. In terms of stiffness S(x) and mass per unit length, the equation and boundary conditions are:

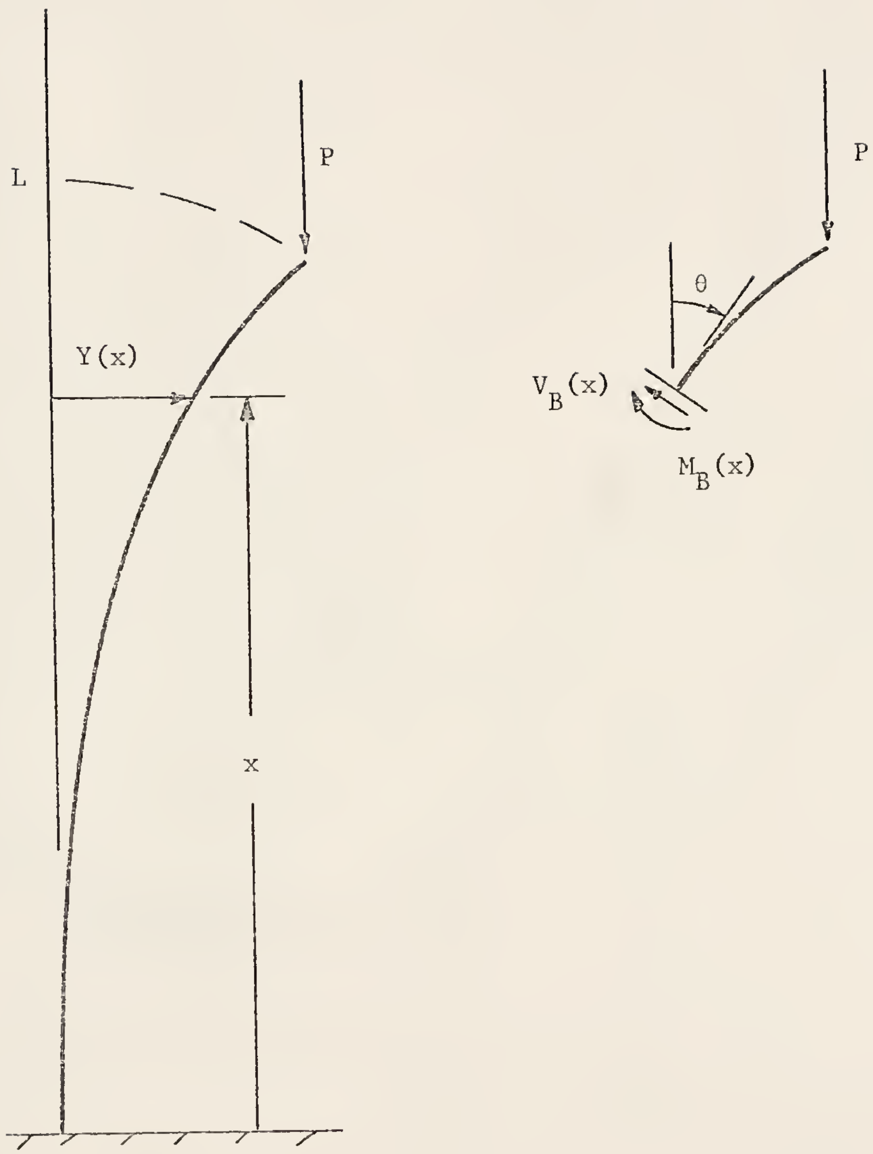


Figure 5.1 Structural Conventions

$$(S(x)Y'')'' + PY'' + g\left(Y' \int_{\xi=x}^{L} M(\xi)d\xi\right)' = 0 \qquad (5.2.1)$$

$$Y(x) = Y'(x) = 0 \qquad\qquad x = 0$$

$$(S(x)Y'') = (S(x)Y'')' + PY' = 0 \qquad x = L$$

Stiffness and mass distributions for the column are

$$S(x) = E(x)I(x)$$

$$M(x) = \rho(x)A(x)$$

where $I(x)$ is the second moment of area about the neutral axis.

It can be shown that assuming similar cross sections provides a relationship between $I(x)$ and $A(x)$. If $f(x)$ represents the dependence of one dimension of the cross section with the position along the beam, then for some reference area A_0,

$$A(x) = A_0 \, f^2(x) \qquad (5.2.2)$$

Choosing a reference area moment I_0 related to A_0 through a radius of gyration k_0 for the cross-sectional shape such that

$$I_0 = k_0^2 A_0$$

it can easily be shown that

$$I(x) = \frac{k_0^2}{A_0} A^2(x)$$

A dimensionless form of the differential equation is obtained by introducing the following dimensionless quantities

$$t = x/L$$

$$\eta = Y/L$$

$$a(x) = A(x)/A_0$$

$$e(x) = E(x)/E_0$$

$$\rho(x) = R(x)/R_0$$

$$m(x) = M(x)/M_0$$

$$s(x) = S(x)/S_0$$

where A_0, E_0, ρ_0 designate some reference value and

$$M_0 = \rho_0 A_0$$

$$S_0 = E_0 I_0$$

and

$$m(x) = a(x)r(x)$$

$$s(x) = a^2(x)e(x)$$

Design variables and the given geometric constraints (5.1.1) are also converted to dimensionless form:

$$a_L \leq a(x) \leq a_U$$

$$e_L \leq e(x) \leq e_U$$

$$r_L \leq \rho(x) \leq r_U$$

where the transformed bounds have been divided by the appropriate reference value. Replacing variables in (5.2.1) by their dimensionless equivalent gives

$$\frac{d^2}{dt^2}\,(s(t)\ddot{\eta}) + \lambda\ddot{\eta} + k\,\frac{d}{dt}\left(\dot{\eta}\int_{\xi=t}^{1} m(\xi)d\xi\right) = 0$$

$$\eta = \dot{\eta} = 0 \qquad\qquad t = 0 \qquad\qquad (5.2.3)$$

$$(s(t)\ddot{\eta}) = \frac{d}{dt}\,(s(t)\ddot{\eta}) + \lambda\dot{\eta} = 0 \qquad\qquad t = 1$$

where

$$\lambda = \frac{PL^2}{E_0 I_0}$$

$$k = \frac{M_0 g L^3}{E_0 I_0}$$

Integrating by parts and using the given boundary conditions, equivalent formulations to (5.2.3) are obtained which are used to convert the governing equation to a state component representation. Equivalent formulations are:

$$(s(t)\ddot{\eta}) = \lambda[\eta(1) - \eta(t)] + k\int_{\xi=t}^{1} m(\xi)[\eta(\xi) - \eta(t)]d\xi$$

$$\frac{d}{dt}\,(s(t)\ddot{\eta}) = -\lambda\dot{\eta} - k\dot{\eta}\int_{\xi=t}^{1} m(\xi)d\xi \qquad\qquad (5.2.4)$$

$$\frac{d}{dt}\left(\dot{\eta}^{-1}\frac{d}{dt}\,(s(t)\ddot{\eta})\right) = km(t)$$

Let

$x_1 = \eta$	Deflection
$x_2 = \dot{\eta}$	Slope
$x_3 = s(t)\ddot{\eta}$	Moment
$x_4 = \dot{\eta}^{-1}\frac{d}{dt}\,(s(t)\ddot{\eta})$	Shear ÷ Slope

then, from this choice of state components and the last of the equations (5.2.4) it follows that

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = x_3/s(t)$$

$$\dot{x}_3 = x_2 x_4$$

$$\dot{x}_4 = km(t)$$

To complete the transformation of the structural problem to an optimal control problem, the design parameters are treated as the components of control $\underline{u}(t)$ where

$$u_1(t) = a(x(t))$$

$$u_2(t) = e(x(t))$$

$$u_3(t) = r(x(t))$$

Hence with the definition of control $\underline{u}(t)$ and the representation of the system's state by a four-dimensional vector, the governing equations are:

$$\begin{aligned} \dot{x}_1 &= x_2 & x_1(0) &= 0 \\ \dot{x}_2 &= x_3/u_1^2 u_2 & x_2(0) &= 0 \\ \dot{x}_3 &= x_2 x_4 & x_3(1) &= 0 \\ \dot{x}_4 &= k u_1 u_3 & x_4(1) &= -\lambda \end{aligned} \tag{5.2.5}$$

To complete the conversion of the structural problem to an optimal control problem, the specified weight condition must be considered. In terms of the specific mass distribution M(x), that condition is

$$W = g \int_0^L M(x)dx$$

or

$$W = \rho_0 A_0 \, gL \int_0^1 m(t)dt$$

This introduces yet another dimensionless parameter to the system in that

$$1 = \mu \int_0^1 u_1(t)u_3(t)dt$$

where

$$\mu = \frac{\rho_0 A_0 gL}{W}$$

With this last development, the buckling problem can be stated with more mathematical precision. Find

$$\underline{u}_{OPT} = \underset{\underline{u} \, \varepsilon \, \Omega}{\text{ARGMAX}} \, [-x_4(1)] \quad , \qquad -x_4(1) = \lambda$$

subject to:

(i)	differential constraints	$\dot{\underline{x}} = \underline{f}(\underline{x},\underline{u})$
(ii)	kinematic boundary conditions	$x_1(0) = 0$
		$x_2(0) = 0$
	natural boundary conditions	$x_3(1) = 0$
		$x_4(1) = -\lambda$
(iii)	hard geometric constraint	$a_L \leq u_1(t) \leq a_U$
(iv)	hard material constraints	$e_L \leq u_2(t) \leq e_U$
		$r_L \leq u_3(t) \leq r_U$
(v)	subsidiary constraint	$\mu \int_0^1 u_1 u_3 dt = 1$

A cursory examination of this problem statement reveals the inherent difficulty of optimal eigenvalue problems. If the eigenvalue λ is the cost functional, the cost functional appears in the governing equation and/or the boundary conditions. For the state components selected, the eigenvalue λ appears only as a boundary condition. However, as stated in (5.2.3), the eigenvalue appears in both the equation and boundary conditions.

5.3 Analysis of the Problem

In the preceding section the governing equation for column buckling is expressed as either a second, third, or fourth order equation with appropriate boundary conditions. State components are then defined from structural quantities, one of which is the shear divided by the slope. Since the slope vanishes at $t = 0$, it must be shown that this state variable, $x_4(t)$, is not indeterminate at the point in question. Moreover, in order to use the theoretical techniques developed in preceding chapters the first eigenvalue (cost) must be expressed as some functional of the state and design parameters.

According to Bolotin (1963, p. 22), conservative systems are described by self-adjoint boundary value problems. Classical elastic stability theory is conventionally restricted to conservative problems in which the buckling load is the fundamental eigenvalue. It is shown in what follows that the column buckling problem is self-adjoint and that the eigenvalue may be obtained via the total potential energy of the system from the Rayleigh quotient. Sufficiency for $x_4(t)$ to be determinate at $t=0$ is that the problem be a self-adjoint eigenvalue problem.

Consider a dependent variable in the third-order representation of the governing equation (5.2.4) and appropriate boundary condition. If

$$z = \dot{\eta}$$

another valid representation of the system is

$$\frac{d}{dt}(s\dot{z}) + (\lambda + k\int_0^1 m(\xi)d\xi)z = 0$$

$$z = 0 \quad , \qquad t = 0$$

$$s(t)\dot{z} = 0 \quad , \qquad t = 1$$

Using the notation of Lovitt (1924), who treats the general problem

$$(pu')' + (q + \lambda r)u = 0$$

$$(puu')\Big|_0^L = 0$$

the corresponding quantities are

$$u \sim z$$

$$x \sim t$$

$$p(x) \sim s(t)$$

$$q(x) \sim k\int_t^1 m(\xi)d\xi$$

$$r(x) \sim 1$$

When written as a linear differential operator L( ) and its adjoint $L^*(\ )$, the operator L( ) is said to be self-adjoint if $L^*(\ ) \equiv L(\ )$.

If the TPBVP is to be self-adjoint for any admissible function ϕ and ψ, then by Green's Formula

$$\int_a^b \{\psi L(\phi) - \phi L^*(\psi)\}\, dx = B(\phi,\psi)\Big|_a^b = 0$$

where $B(\phi,\psi)$ is the bilinear concomitant of $L(\ )$. In terms of a general second order equation

$$L(\phi) = p_1\phi'' + p_2\phi' + (q + \lambda r)\phi$$

$$L^*(\psi) = (p_1\psi)'' - (p_2\psi)' + (q + \lambda r)\psi$$

The necessary and sufficient conditions that $L(\ )$ be self-adjoint is that $p_2 = p_1'$. If $L(\ )$ is a self-adjoint operator and the bilinear concomitant vanishes, the problem is said to be self-adjoint. For the general operator,

$$B(\phi,\psi)\Big|_a^b = p_1(\phi'\psi - \phi\psi')$$

or in terms of Lovitt's equation, which obviously has a self-adjoint operator,

$$B(\phi,\psi)\Big|_a^b = p(\phi'\psi - \phi\psi')\Big|_0^1$$

it is seen that the problem is self-adjoint if the boundary conditions of the adjoint and given systems are identical.

There is a well-developed theory associated with self-adjoint eigenvalue problems, where the eigenvalue problem itself is the result of minimizing an energy functional. Along this line, Lovitt shows that under the requirements that $p(x) > 0$ and is piecewise smooth in $(0,1)$, and that $q(x)$ is a well-behaved function which is finite everywhere on $(0,1)$, there are an infinite number of real positive eigenvalues which may be arranged as

$$0 < \lambda_1 < \lambda_2 < \ldots < \lambda_n < \ldots$$

If p(x) and q(x) are specified functions which satisfy the above conditions, the fundamental eigenvalue is given by

$$\lambda_1 = \underset{u(x)}{\text{Min}} \; [D(u)]$$

$$D(u) = \int_0^1 \{p(x)(u')^2 - q(x)u^2\} \, dx$$

where

$$u \varepsilon C^2$$

$$puu'\Big|_0^1 = 0$$

and

$$\int_0^1 r(x)u^2 dx = 1$$

Lovitt shows for $u \varepsilon C^2$ and p(x), q(x), r(x) bounded, that λ_1 is finite.

Higher ordered eigenvalues are obtained by using the orthogonality property of eigenfunctions as additional subsidiary conditions. For example,

$$\lambda_n = \underset{u_n(x)}{\text{Min}} \; [D(u_n)]$$

$$D(u_n) = \int_0^1 \{p(x)(u_n')^2 - q(x)u_n^2\} \, dx$$

for

$$u_n \varepsilon C^2$$

$$pu_n u_n' \Big|_0^1 = 0$$

$$\int_0^1 r(x) u_n^2 \, dx = 1$$

and

$$\int_0^1 r(x) u_n u_i dx = 0 \qquad i = 1, \ldots, n-1$$

The immediate use of the self-adjoint property of the column buckling problem is that the fundamental eigenvalue is finite. This is used to prove that $x_4(0)$ is determinate. Recall the definition

$$x_4 = \dot{\eta}^{-1} \frac{d}{dt} (s\ddot{\eta})$$

From the third-order formulation of the governing equation (5.2.4)

$$\dot{\eta}^{-1} \frac{d}{dt} (s\ddot{\eta}) = - \lambda - k \int_t^1 m(\xi) d\xi$$

such that

$$x_4(0) = - \lambda - k \int_0^1 m(\xi) d\xi = - \beta$$

where β is a positive constant to be determined. By substituting the specified weight subsidiary condition for the integral

$$x_4(0) = - (\lambda + k/\mu) = - \beta$$

Both k and μ are specified, finite parameters of the system. If λ is finite then $x_4(0)$ is also. In fact, the buckling load λ is the

fundamental eigenvalue λ_1 which is shown by Lovitt to be finite for the conditions prescribed. Hence, $x_4(0)$ is determinate.

In the literature of classical elastic stability problems, the eigenvalue to be optimized is expressed as a Rayleigh quotient. This quotient can be found from the total potential energy. For the general equation of Lovitt, the quotient is

$$\lambda = \frac{\int_0^1 \{p(u')^2 - qu^2\}\, dx}{\int_0^1 ru^2 dx}$$

or in terms of the column buckling parameters and components (5.2.5)

$$\lambda = \frac{\int_0^1 \{x_3^2/s(t) - k \int_t^1 m(\xi)d\xi\ x_2^2\}\, dt}{\int_0^1 x_2^2\, dt}$$

On reversing the order of integraion for the double integral contained in the numerator, and introducing design parameters, the Rayleigh quotient becomes

$$\lambda = \frac{\int_0^1 \{x_3^2/u_1^2u_2 - ku_1u_3 \int_0^t x_2^2(\xi)d\xi\}\, dt}{\int_0^1 x_2^2\, dt}$$

The normalization condition of Lovitt requires that the quotient's denominator equals unity for all admissible solutions, or

$$\int_0^1 x_2^2(t)dt = 1$$

This particular normalization is selected in order to obtain the simplest expression for the functional λ. Also, recall that

$$x_4(0) = -(\lambda + k/\mu)$$

and

$$x_4(1) = -\lambda$$

It is seen from these boundary conditions and the definition $\dot{x}_4 = ku_1u_3$ that the subsidiary weight condition can be written as a mixed boundary condition on $x_4(t)$:

$$x_4(1) - x_4(0) = k/\mu$$

This, together with converting the normalization condition to another state component with boundary conditions specified at both ends, transforms the problem to:

$$\lambda_{Max} = \underset{\underline{u} \in \Omega}{\mathrm{Max}} \{J[\underline{u}]\}$$

$$J[\underline{u}] = \int_0^1 \{x_3^2/u_1^2u_2 - ku_1u_3x_5\}\, dt$$

subject to

$$\begin{aligned}
\dot{x}_1 &= x_2 & \qquad x_1(0) &= 0 \\
\dot{x}_2 &= x_3/u_1^2u_2 & x_2(0) &= 0 \\
\dot{x}_3 &= x_2x_4 & x_3(1) &= 0 \\
\dot{x}_4 &= ku_1u_3 & x_4(1) - x_4(0) &= k/\mu \\
\dot{x}_5 &= x_2^2 & x_5(0) &= 0 \\
 & & x_5(1) &= 1
\end{aligned}$$

where $\underline{u} \varepsilon \Omega$ means that $\underline{u} \varepsilon C^2$ and satisfies the geometric control inequalities. For reasons as yet undetermined, the quasilinearization algorithm did not converge when the latter two state boundary conditions were used, even though they belong to the class of problems to which the algorithm is theoretically applicable. This might be due solely to the mixed boundary condition being dependent on the value of the cost functional being minimized.

On failing to obtain a solution by minimization of the Rayleigh quotient, the next attempt was suggested by the state boundary conditions. From the condition

$$x_4(1) = -\lambda$$

and definition of λ, whereby $\lambda > 0$, it follows that

$$\underset{\underline{u}}{\text{Min}} \; [x_4(1)] \equiv \underset{\underline{u}}{\text{Max}} \; [\lambda]$$

Thus, the problem can be cast as a Mayer type of optimal control problem:

$$\lambda_{\text{Max}} = \underset{\underline{u} \varepsilon \Omega}{\text{Min}} \; [x_4(1)]$$

$$\dot{x}_1 = x_2 \qquad\qquad x_1(0) = 0$$

$$\dot{x}_2 = x_3/u_1^2 u_2 \qquad\qquad x_2(0) = 0$$

$$\dot{x}_3 = x_2 x_4 \qquad\qquad x_3(1) = 0$$

$$\dot{x}_4 = k u_1 u_3 \qquad\qquad x_4(0) = -\beta$$

where β is a positive constant to be determined. The quasilinearization algorithm did not converge for this formulation of the problem also, again for undetermined reasons.

Consider from the choice of state variables that

$$\dot{x}_4 = km(t) \qquad \text{and} \qquad x_4(1) = -\lambda$$

In an earlier derivation, β was defined such that

$$x_4(0) = -\lambda - k\int_0^1 m(t)\,dt = -\beta < 0$$

In another form,

$$-\beta + k\int_0^1 m(t)\,dt = -\lambda$$

$$(\text{Constant}) + (\text{Weight}) = -\ (\text{Load})$$

or in terms of the weight

$$\lambda = \beta - kW$$

Therefore, if the weight W is minimized subject to constraints, this is the same as maximizing λ, the load, subject to the same constraints. Notice that constant β is an undetermined boundary condition, i.e., a free condition, so that the corresponding adjoint variable's boundary condition is known.

At this point the results of a doctoral dissertation--Salinas (1968)--were used in another formulation of the same problem. Salinas considered the application of the energy method to self-adjoint problems. He showed that the following problems are identical:

(i) Maximum load, specified weight

$$\underline{u}_{OPT} = \underset{\underline{u}\,\varepsilon\,\Omega}{\text{ARGMAX}} \ \{J_P[\underline{u}]\}$$

J_P = the buckling load cost functional

J_W = a constant, the weight

(ii) Minimum weight, specified load

$$\underline{u}_{OPT} = \underset{\underline{u}\,\varepsilon\,\Omega}{\text{ARGMIN}} \{J_W\ [\underline{u}]\}$$

J_W = the weight cost functional

J_P = a constant, the load

Both problems are subject to the same differential constraints, boundary conditions, and control admissibility constraints. A concise way of stating the equivalence is

$$\underline{u}_{OPT} = \underset{\underline{u}\,\varepsilon\,\Omega}{\text{ARGMAX}} \ \{J_P\ [\underline{u}] : J_W \text{ specified}\}$$

$$= \underset{\underline{u}\,\varepsilon\,\Omega}{\text{ARGMIN}} \ \{J_W\ [\underline{u}] : J_P \text{ specified}\}$$

The equivalence of these two problems is used to state the column buckling problem as

$$\underline{u}_{OPT} = \underset{\underline{u}\,\varepsilon\,\Omega}{\text{ARGMIN}} \ [J_W]$$

$$J_W = \mu \int_0^1 u_1 u_3 \, dt$$

$$\dot{x}_1 = x_2 \qquad\qquad x_1(0) = 0$$

$$\dot{x}_2 = x_3/u_1^2 u_2 \qquad\qquad x_2(0) = 0$$

$$\dot{x}_3 = x_2 x_4 \qquad\qquad x_3(1) = 0$$

$$\dot{x}_4 = k u_1 u_3 \qquad\qquad x_4(1) = -\lambda_s$$

where the subscript s on the eigenvalue indicates a specified load. With reference to the statement of the problem as a Mayer type, notice that

$$J_W = \frac{\mu}{k} [x_4(1) - x_4(0)] \rightarrow \underset{\underline{u} \, \varepsilon \, \Omega}{\text{Min}} \; [x_4(1)]$$

with $x_4(0) = -\beta$, a free boundary condition. Furthermore, since

$$x_4(1) = -\lambda \quad \text{and} \quad \lambda \equiv J_P$$

observe that

$$\frac{k}{\mu} J_W = \beta - J_P$$

Admissibility of control ($\underline{u} \, \varepsilon \, \Omega$) requires that $\underline{u}(t)$ is piecewise continuous through the second derivative, and that the control inequality constraints are not violated. For every value λ_s assigned a priori to the eigenvalue λ there is a solution $(\underline{u}_{W_{OPT}}, \underline{x}_{W_{OPT}})$ that minimizes the weight with respect to $\underline{u} \, \varepsilon \, \Omega$ and satisfies the eigenvalue problem. However, an arbitrary λ_s need not satisfy the specified weight condition and in general does not. To fulfill the given total weight requirement the value of λ_s was adjusted such that

$$J_W \, [\underline{u}_{W_{OPT}}] = \mu \int_0^1 u_1 u_3 dt = 1$$

When the cost is not only a minimum but is also equal to unity, all conditions of the buckling problem as stated in section 5.2 are satisfied. By Salinas' theorem the λ_s causing this satisfaction is the maximum lowest eigenvalue, i.e., the maximum buckling load. This effect is illustrated in Figure 5.2 and is mathematically stated by

$$\lambda_{Max} = \{J_P[\underline{u}_{W_{OPT}}] : J_W[\underline{u}_{W_{OPT}}] = 1\}$$

where

$$\underline{u}_{W_{OPT}} = \underset{\underline{u} \, \varepsilon \, \Omega}{\text{ARGMIN}} \, \{J_W[\underline{u}] : \lambda = \lambda_s\}$$

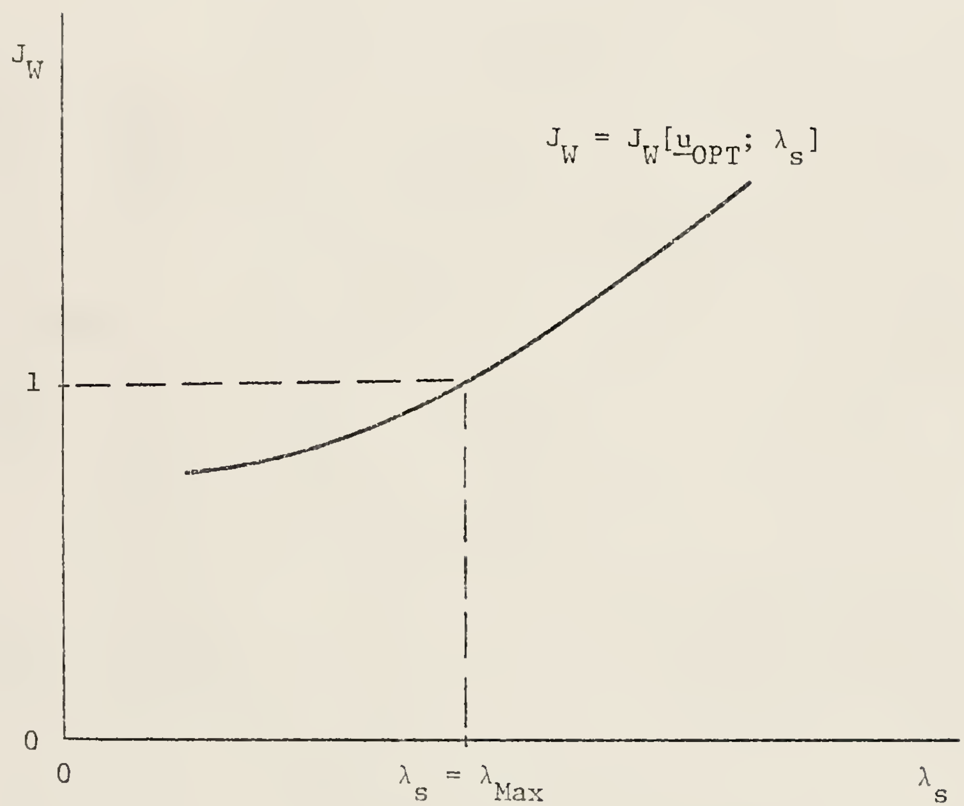


Figure 5.2 Relationship Between Optimal Eigenvalue and Minimum Weight Cost Functional

5.4 Application of the Maximum Principle

In the preceding analysis difficulties associated with the optimal eigenvalue problem are discussed. Circumvention of such difficulties is described for those problems that are self-adjoint. It is shown that the column buckling belongs to that class of problems. However, before the theoretical developments presented in the third chapter may be applied, the particular forms of parameters and functions must be identified. Accordingly, for the optimization of the column buckling load

$$\underline{x} \sim (x_1\ x_2\ x_3\ x_4)^T$$

$$\underline{u} \sim (u_1\ u_2\ u_3)^T$$

$$J \sim J_W = \int_0^1 L_0(\underline{x},\underline{u})\,dt$$

$$L_0(\underline{x},\underline{u}) = \mu u_1 u_3$$

$$\underline{f}(\underline{x},\underline{u}) = \begin{cases} x_2 \\ x_3/u_1^2 u_2 \\ x_2 x_4 \\ k u_1 u_3 \end{cases}$$

$$\text{Boundary Conditions} \sim \begin{cases} x_1(0) = 0 \\ x_2(0) = 0 \\ x_3(1) = 0 \\ x_4(1) = -\lambda_s \end{cases}$$

$$\phi_\ell(\underline{x},\underline{u}) \sim \begin{cases} -\ (a_U-u_1)(u_1-a_L) \\ -\ (e_U-u_2)(u_1-e_L) \\ -\ (r_U-u_3)(u_3-r_L) \end{cases} ,\ \mu_\ell \begin{cases} \mu_1 \\ \mu_2 \\ \mu_3 \end{cases} \tag{5.4.1}$$

where the maximum value of the λs is that value such that

$$J_W = \mu \int_0^1 u_1 u_3 dt = 1$$

Based upon these particular forms for the general parameters, the variational Hamiltonian is

$$H = -\ \{p_1x_2 + p_2x_3/u_1^2u_2 + p_3x_2x_4 + p_4ku_1u_3\} + \mu u_1u_3$$

Adjoint variables $\underline{p}(t)$ are defined from the Hamiltonian H^* which contains the constraint functions. Since the constraint functions ϕ_ℓ are independent of $\underline{x}$ for the case of only geometric constraints

$$\dot{p} = -\ H^*_{\underline{x}} \equiv H_{\underline{x}}$$

From this result and from the state boundary conditions it follows that

$$\begin{aligned} \dot{p}_1 &= 0 & p_1(1) &= 0 \\ \dot{p}_2 &= -p_1x_2 - p_3x_4 & p_2(1) &= 0 \\ \dot{p}_3 &= -p_2/u_1^2u_2 & p_3(0) &= 0 \\ \dot{p}_4 &= -p_3x_2 & p_4(0) &= 0 \end{aligned}$$

Variable $p_1(t)$ may be integrated by inspection:

$$p_1(t) = 0 \qquad \text{for all} \qquad t \geq 0$$

which immediately reduces the number of adjoint variables to be determined and simplifies the Hamiltonian as well. Thus,

$$H = -\{p_3x_2x_4 + p_2x_3/u_1^2u_1 + (kp_4-\mu)u_1u_3\}$$

where

$$\dot{p}_2 = -p_3x_4 \qquad p_2(1) = 0$$

$$\dot{p}_3 = -p_2/u_1^2u_2 \qquad p_3(0) = 0$$

$$\dot{p}_4 = -p_3x_2 \qquad p_4(0) = 0$$

Optimal control as determined by the PMP stated in (3.1.8) is

$$\underline{u}_{OPT} = \underset{\underline{u}\,\varepsilon\,\Omega}{\mathrm{ARGMIN}}\ [H(\underline{x}_{OPT},\underline{u},\underline{p})] \tag{5.4.2}$$

where the admissibility restriction in Pontryagin's derivation requires only that $\underline{u}(t)$ be piecewise continuous. Applying Salinas' theorem further restricts the control to $\underline{u}\ \varepsilon\ C^2$. Considering for the moment that no further constraints on the control exist, the unconstrained $\underline{u}_{OPT}$ is that choice of $(u_1(t),\ u_2(t),\ u_3(t))$ which minimizes the variational Hamiltonian H. Recognizing that the physical quantities represented by $\underline{u}$ have meaning only if they are positive, assume this to be the case. With non-negative controls, the state equations and boundary conditions jointly imply that

$$x_i(t) \geq 0 \qquad i = 1,2,3$$

$$x_4(t) < 0$$

Furthermore, by means of (5.4.2) and the expression for H

$$(p_2x_3) > 0 \rightarrow u_2 = 0$$

$$(p_2x_3) < 0 \rightarrow u_2 = \infty$$

and

$$(kp_4-\mu) > 0 \rightarrow u_3 = \infty$$

$$(kp_4-\mu) < 0 \rightarrow u_3 = 0$$

Control u_1 remains to be determined.

Assuming that u_2, u_3 are known quantities, then u_1 must be chosen independently to minimize the variational Hamiltonian. In that context

$$H_{u_1} = - \{-2p_2x_3/u_1^3u_2 + (kp_4-\mu)u_3\} = 0$$

$$H_{u_1u_1} = - \{6p_2x_3/u_1^4u_2\} > 0$$

The unconstrained optimal control variable u_1^* is defined to satisfy the first condition; the second condition indicates that $p_2(t)$ must be negative for $H(\underline{x}_{OPT},\underline{u}_{OPT},\underline{p})$ to be a minimum. So,

$$u_1^* = \left\{\frac{2p_2x_3}{(kp_4-\mu)u_2u_3}\right\}^{1/3}$$

and

$$p_2(t) < 0$$

which leads to the following argument on the signs of $p_i(t)$:

$$\dot{p}_2 > 0 \leftarrow p_2(t) < 0 \quad \text{and} \quad p_2(1) = 0$$

$$p_3(t) > 0 \leftarrow \dot{p}_2 = -p_3x_4 \quad \text{and} \quad x_4(t) < 0$$

$$\dot{p}_4 < 0 \leftarrow \dot{p}_4 = -p_3x_2 \quad \text{and} \quad x_2(t) > 0$$

$$p_4(t) < 0 \leftarrow \dot{p}_4 < 0 \quad \text{and} \quad p_4(0) = 0$$

It is noted that with these signs if $(kp_4-\mu) < 0$ the optimal unconstrained control is

$$\underline{u}_{OPT} = \begin{Bmatrix} u_1^* \\ \infty \\ 0 \end{Bmatrix}$$

where the existence of u_1^* is discussed in the next section. What is important is that the solution requires a material of infinite strength and zero density. Some structural analysts commonly refer to this hypothetical material as Bolognium.

In the course of an actual design process, control bounds corresponding to the constraint functions are specified. On the basis of the preceding analysis with given bounds on controls, optimal $u_2(t)$ and $u_3(t)$ are bang-bang controls defined by

$$u_2(t) = \tfrac{1}{2}(e_L+e_U) + \tfrac{1}{2}(e_L-e_U)\ \text{SGN}\ (p_2x_3) \tag{5.4.3}$$

$$u_3(t) = \tfrac{1}{2}(r_L+r_U) - \tfrac{1}{2}(r_L-r_U)\ \text{SGN}\ (kp_4-\mu) \tag{5.4.4}$$

where for an arbitrary argument z of function SGN ( )

$$\text{SGN}\ (z) \triangleq \begin{cases} +1 & , \quad z > 0 \\ 0 & , \quad z = 0 \\ -1 & , \quad z < 0 \end{cases}$$

Depending upon the magnitude of u_1^* vis-à-vis a_L and a_U, there are three possibilities for $\underline{u}_{OPT}$ at each point $(t,\underline{x}(t),\underline{p}(t))$:

$$\text{(i)} \quad u_1^* \leq a_L \qquad \underline{u}_{OPT} = \begin{Bmatrix} a_L \\ u_2 \\ u_3 \end{Bmatrix}$$

$$\text{(ii)} \quad a_L < u_1^* < a_U \qquad \underline{u}_{OPT} = \begin{Bmatrix} u_1^* \\ u_2 \\ u_3 \end{Bmatrix}$$

$$\text{(iii)} \quad a_U \leq u_1^* \qquad \underline{u}_{OPT} = \begin{Bmatrix} a_U \\ u_2 \\ u_3 \end{Bmatrix}$$

That class of problems having no mixed constraints requires computing the Lagrangian multiplier functions $\underline{\mu}(t)$ of (3.3.4) only as a check. Each individual multiplier function must be non-negative for all $0 \leq t \leq 1$. A positive value in a mathematical programming application indicates that the usable, feasible direction is attempting to leave an active constraint surface. With the similarity between this and PMP demonstrated in Chapter III, the existence of a negative multiplier function indicates a similar inconsistency in active constraints and the constrained direction of steepest descent of H with respect to $\underline{u}$.

In the usual application of PMP to a problem, the constraints are adjoined to the cost function and another functional defined:

$$H^* = - H - \underline{\mu}^T(t)\ \underline{\phi}(\underline{x},\underline{u})$$

Each constraint $\phi_\ell(\underline{x},\underline{u})$ which is active such that

$$\phi_\ell(\underline{x},\underline{u}) = 0$$

is solved for a component of control. The resulting non-zero multiplier function $\mu_\ell(t)$ must be determined from the necessary condition

$$H^*_{\underline{u}} = 0$$

This $\mu_\ell(t)$ is used in turn to determine the effect upon the system from the solution's lying on a mixed constraint surface which by definition depends upon the state. This effect is manifested by the adjoint differential equation $\dot{\underline{p}} = - H^*_{\underline{x}}$ which can be written as

$$\dot{\underline{p}} = - (H_{\underline{x}} + \underline{\phi}^T_{\underline{x}}\mu)$$

If constraints ϕ_ℓ depend only upon the control, the second term is unnecessary because $\underline{\phi}^T_{\underline{x}} = 0$.

Such is the case of the column buckling problem possessing only control bounds. For this problem, whenever the constraints are active the multiplier functions determined by the above procedure are

$$\mu_1(t) = \frac{-2p_2x_3/u_1^3u_2 + (kp_4-\mu)u_3}{2u_1 - (a_L + a_u)} \tag{5.4.5}$$

$$\mu_2(t) = \frac{-2p_2x_3/u_1^2u_2^2}{2u_2 - (e_u+e_L)} \tag{5.4.6}$$

$$\mu_3(t) = \frac{(kp_4-\mu)u_1}{2u_3 - (r_u+r_L)} \tag{5.4.7}$$

These same parameters will now be determined from the explicit formulation of $\underline{\mu}(t)$ derived in Chapter III.

Recall that where I_A denotes the set of active constraints which are r in number:

$$I_A = \{\alpha_1, \alpha_2, \ldots, \alpha_r\}$$

$$\underline{\phi}^T = \{\phi_{\alpha_1} \; \phi_{\alpha_2} \; \cdots \; \phi_{\alpha_r}\}$$

$$\underline{\phi}_{\underline{u}}^T = [\phi_{ij}] = \left[\frac{\partial \phi_{\alpha_j}}{\partial u_i}\right] \qquad \begin{array}{l} i = 1,\ldots,m \\ j = 1,\ldots,r \end{array}$$

and that in terms of $\underline{\phi}$ and H, $\underline{\mu}(t)$ is defined as

$$\underline{\mu}(t) = -\left(\underline{\phi}_{\underline{u}^T} \, \underline{\phi}_{\underline{u}}^T\right)^{-1} \underline{\phi}_{\underline{u}^T} \, H_{\underline{u}} \tag{3.3.4}$$

For the column buckling problem subject to constraints (5.4.1), in order to calculate $\underline{\mu}$ the following quantities must be defined:

$$\phi_{1,\underline{u}} = \begin{Bmatrix} \phi_{1,1} \\ 0 \\ 0 \end{Bmatrix} \qquad \phi_{2,\underline{u}} = \begin{Bmatrix} 0 \\ \phi_{2,2} \\ 0 \end{Bmatrix} \qquad \phi_{3,\underline{u}} = \begin{Bmatrix} 0 \\ 0 \\ \phi_{3,3} \end{Bmatrix}$$

$$H_{\underline{u}} = \begin{Bmatrix} H_{u_1} \\ H_{u_2} \\ H_{u_3} \end{Bmatrix} = -\begin{Bmatrix} -2p_2x_3/u_1^3u_2 + (kp_4-\mu)u_3 \\ -2p_2x_3/u_1^2u_2^2 + 0 \\ 0 + (kp_4-\mu)u_1 \end{Bmatrix}$$

with

$$\phi_{1,1} = 2u_1 - (a_U + a_L)$$

$$\phi_{2,2} = 2u_2 - (e_U + e_L)$$

$$\phi_{3,3} = 2u_3 - (r_U + r_L)$$

Consider first the case where u_1 is not on a control boundary. Only the latter two constraints are active since u_2 and u_3 are bang-bang controls, thus $\mu_1 = 0$ and both μ_2 and μ_3 are determined by (3.3.4). From the preceding paragraph's definitions

$$r = 2 \rightarrow I_A = \{\alpha_1, \alpha_2\} = \{2,3\}$$

$$\underline{\phi} = \begin{Bmatrix} \phi_2 \\ \phi_3 \end{Bmatrix} \quad , \quad \underline{\mu} = \begin{Bmatrix} \mu_2 \\ \mu_3 \end{Bmatrix}$$

$$(\underline{\phi}_{\underline{u}} {}^T \underline{\phi}_{\underline{u}} {}^T) = \begin{bmatrix} 0 & \phi_{2,2} & 0 \\ 0 & 0 & \phi_{3,3} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ \phi_{2,2} & 0 \\ 0 & \phi_{3,3} \end{bmatrix} = \begin{bmatrix} \phi_{2,2}^2 & 0 \\ 0 & \phi_{3,3}^2 \end{bmatrix}$$

$$(\underline{\phi}_{\underline{u}} {}^T \underline{\phi}_{\underline{u}} {}^T)^{-1} = \begin{bmatrix} \phi_{2,2}^{-2} & 0 \\ 0 & \phi_{3,3}^{-2} \end{bmatrix}$$

$$\underline{\mu} = - \begin{bmatrix} \phi_{2,2}^{-2} & 0 \\ 0 & \phi_{3,3}^{-2} \end{bmatrix} \begin{bmatrix} 0 & \phi_{2,2} & 0 \\ 0 & 0 & \phi_{3,3} \end{bmatrix} \begin{Bmatrix} H_{u_1} \\ H_{u_2} \\ H_{u_3} \end{Bmatrix}$$

$$= - \begin{Bmatrix} H_{u_2}/\phi_{2,2} \\ H_{u_3}/\phi_{3,3} \end{Bmatrix}$$

Hence,

$$\mu_2 = \frac{-p_2 x_3/u_1^2 u_2^2}{2u_L - (e_U+e_L)}$$

$$\mu_3 = \frac{(kp_4-\mu)u_1}{2u_3 - (r_U+r_L)}$$

which are identical to the expressions (5.4.6) and (5.4.7) obtained by the usual method.

Whenever the ϕ_1 constraint is active, u_1 is prescribed and μ_1 is nonzero. If the derivation in the preceding paragraph is expanded to include this additional constraint, the resulting expression for μ_1 is identical to (5.4.5).

The result of this analysis is a nonlinear TPBVP whose solution is the desired optimal solution:

$$\begin{array}{ll}
\dot{x}_1 = x_2 & x_1(0) = 0 \\
\dot{x}_2 = x_3/u_1^2 u_2 & x_2(0) = 0 \\
\dot{x}_3 = x_2 x_4 & x_3(1) = 0 \\
\dot{x}_4 = k u_1 u_3 & x_4(1) = -\lambda_s \\
\dot{p}_2 = -p_3 x_4 & p_2(1) = 0 \\
\dot{p}_3 = -p_2/u_1^2 u_2 & p_3(0) = 0 \\
\dot{p}_4 = -p_3 x_2 & p_4(0) = 0
\end{array}$$

$$\underline{u}_{OPT} = \underset{\underline{u}\,\varepsilon\,\Omega}{\mathrm{ARGMIN}}\ [H(\underline{x}_{OPT}, \underline{u}, \underline{p})]$$

$$H = -\{p_3 x_2 x_4 + p_2 x_3/u_1^2 u_2 + (kp_4-\mu)u_1 u_3\}$$

$$\lambda_{Max} = \{J_P[\underline{u}_{OPT}] : J_W[\underline{u}_{OPT}] = 1\}$$

where $\underline{u} \in \Omega$ requires that $\underline{u}$ satisfy constraints (5.4.1) and Lagrangian multiplier functions given by (5.4.5) through (5.4.7) must be non-negative. A solution to this problem is obtained by the method of quasilinearization; a subroutine listing of the equations required by the IBM program is given in Appendix C.

One additional preliminary item must be attended to before presenting results. In order to provide solutions subject to comparison standards, an eigenfunction normalization must be prescribed. Most Rayleigh quotient problems in the open literature normalize the eigenfunction such that the denominator of the quotient is always equal to unity. For the column buckling problem this is equivalent to

$$\int_0^1 x_2^2(t)\, dt = 1 \tag{5.4.8}$$

The unmodified results of solution via quasilinearization do not satisfy this condition; instead,

$$\int_0^1 x_2^2(t)\, dt = A^2$$

where A is some constant. A transformation which satisfies normalization (5.4.8) such that the differential equations, boundary conditions, and optimal control are unaffected by the transformation, is given by

$$\begin{array}{ll} x_1 \to x_1/A & - \\ x_2 \to x_2/A & p_2 \to Ap_2 \\ x_3 \to x_3/A & p_3 \to Ap_3 \\ x_4 \to x_4 & p_4 \to p_4 \end{array}$$

With a specified normalization of the eigenfunction, the state solution $\underline{x}_{OPT}$ may be compared to any other similar solution for which the normalization is known. All subsequent results are normalized as above to satisfy (5.4.8).

5.5 Results: Geometric Control Constraints

Three dimensionless parameters of the system are derived in section 5.3. The first is λ, the eigenvalue to be maximized. The second is k, which is a relative measure of the weight of a uniform column to its stiffness--how strongly inclusion of the weight affects the solution. The third parameter, μ, is a measure of the weight of a uniform column described by the reference conditions relative to the weight of the column having distributed geometry and properties. Values of k and μ describe a particular problem statement and must be specified a priori.

Since the present discussion is directed towards technique, as opposed to results for particular geometries, loading conditions, and materials, a single value of both μ and k is used for all cases. Most of the literature compares the optimal eigenvalue to that obtained for a uniform cylinder of identical weight. This corresponds to a value of unity for μ, selected so that the results may be compared. Parameter k involves both material and geometric properties. For convenience the material selected is structural steel and the Euler buckling slimness ratio (L/k_0) value of 60 is assumed. With these assumptions the values used throughout are

$$\mu = 1 \qquad\qquad k = .012$$

At this point a comment upon the effect that the magnitude of k has upon the nature of $\underline{u}_{OPT}(t)$ is appropriate. In the last section it was shown that u_2 and u_3 are bang-bang controls dependent upon the sign of switching functions (p_2x_3) and $(kp_4-\mu)$, respectively. Intuitively one expects the strongest, least dense material to be optimal, corresponding to

$$(p_2x_3) < 0 \rightarrow u_2 = e_U$$

$$(kp_4-\mu) < 0 \rightarrow u_3 = r_L$$

Also shown in the last section is that for Hamiltonian H to be minimized by an unconstrained value of $u_1 = u_1^*$ at some time, it is necessary that $p_2(t) < 0$. Note that if this does not occur somewhere in $0 \leq t \leq 1$ the control is uniform throughout, and the system described is a uniform column; unless $\lambda_s = \pi^2/4$ the solution of the state is the trivial solution.† A direct consequence of p_2 being non-positive is that $p_4(t)$ is also. Hence any positive value of k is sufficient to make $(kp_4-\mu) < 0$, so that the minimum density is required.

This insignificant requirement that a nontrivial, optimal solution to the buckling problem exist provides an unusual result. When the variable signs required for a nontrivial solution are used in the PMP expressions (5.4.3) and (5.4.4), the intuitive solution is shown mathematically to be the optimal solution. It is for this reason that all subsequent cases were run with control bound values

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†Coding errors produced precisely this case, and indeed, to within the numerical limits of the computer the algorithm converged to the trivial solution.

$$e_U = 1.0 \qquad r_U = 1.1$$

$$e_L = 0.8 \qquad r_L = 1.0$$

Unity values were selected to simplify results; nonunity control bound values were included to reveal unforeseen switching in u_2 and u_3. None were encountered. Effectively, the problem as stated has a single control component $u_1(t) \equiv a(x/L)$.

To illustrate the operation of the quasilinearization algorithm, detailed results are presented for one case. Data for this case are:

$$\begin{aligned} a_U &\to \infty \\ a_L &= 0.9 \end{aligned} \tag{5.5.1}$$

Just as in the cantilever beam example, the initial guess is taken from the uniform solution and must satisfy only the boundary conditions. Initial guesses are in terms of

$$\left.\begin{aligned} C(t) &= \cos\left(\tfrac{1}{2}\pi t\right) \\ S(t) &= \sin\left(\tfrac{1}{2}\pi t\right) \end{aligned}\right. \qquad 0 \le t \le 1$$

such that

$$x_1(t) = \frac{4}{\pi}\,(1-C(t))$$

$$x_2(t) = 2S(t)$$

$$x_3(t) = C(t)$$

$$x_4(t) = -\lambda_s - (1-t)k/\mu$$

$$x_5(t) = -\frac{1}{2}\,C(t)$$

$$x_6(t) = \frac{1}{\pi}\,S(t)$$

$$x_7(t) = -\frac{1}{\pi}\,S$$

Given this initial guess, the value of λ_s which satisfies the weight condition is $\lambda_{Max} = 2.75$. This is demonstrated by the following values of λ_s and J_W which represent specific points on a curve like that of Figure 5.2:

λ_s	J_W
2.74108	0.998819
2.74396	0.999210
2.75000	1.000000

For $\lambda_{Max} = \lambda_s$ such that $J_W = 1$ the convergence is measured in terms of ERROR as defined in Section 4.4, and shown in the following tabulation:

Iteration	ERROR	Component (Position)	Cost J_W	Cost J_P
0	$.696*10^{-2}$	$x_2(0)$	1.00000	2.75
1	$.151*10^{-2}$	$x_2(.630)$	1.00017	2.75
2	$.128*10^{-4}$	$x_2(.555)$	1.00003	2.75
3	$.250*10^{-12}$	$x_2(.550)$	1.00000	2.75

Deflection of the centerline is shown in Figure 5.3 which contains three curves: the initial guess, a plot of subsequent iterates (coincident for the scale used), and the normalized solution. The optimal control is displayed in Figure 5.4. Control functions for iterations subsequent to the initial guess plot coincident to a single curve. All non-zero components of the state and adjoint variables are shown in Figure 5.5. Shear (represented by the product x_2x_4) is also included to give a complete set of structural variables.

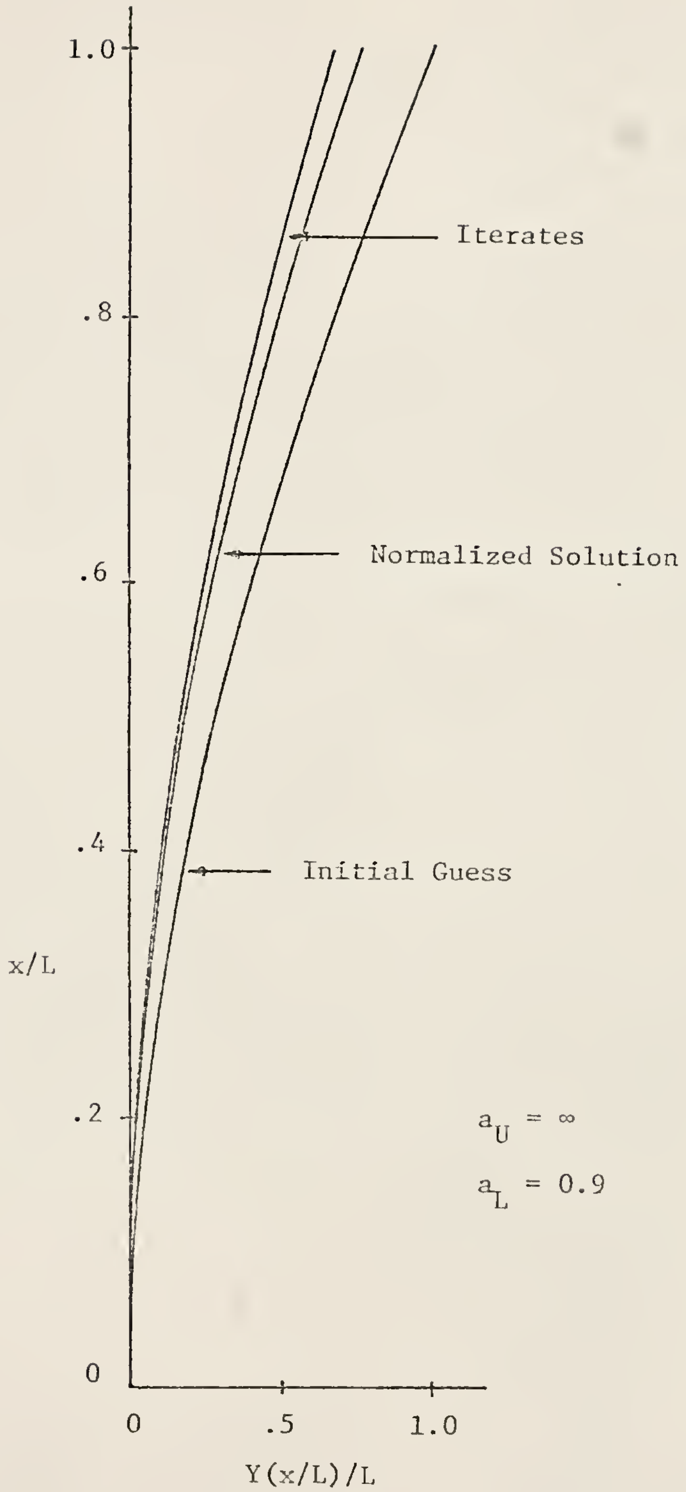

Figure 5.3 Typical Deflection of Centerline

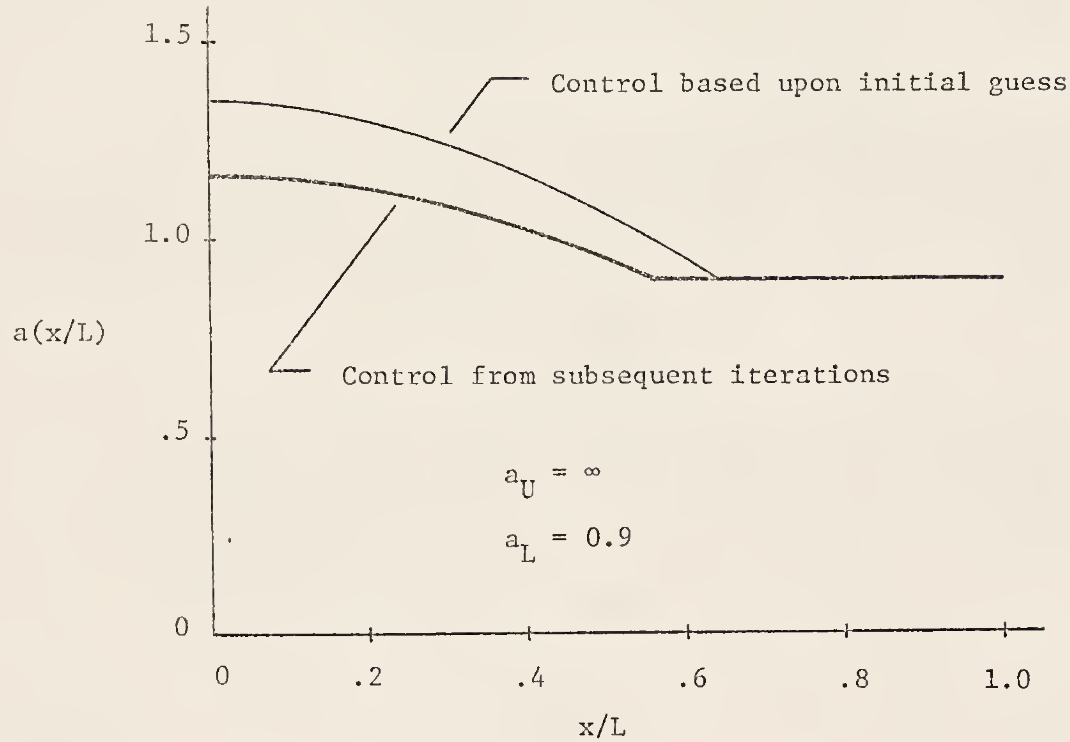

Figure 5.4 Typical Optimal Control Function

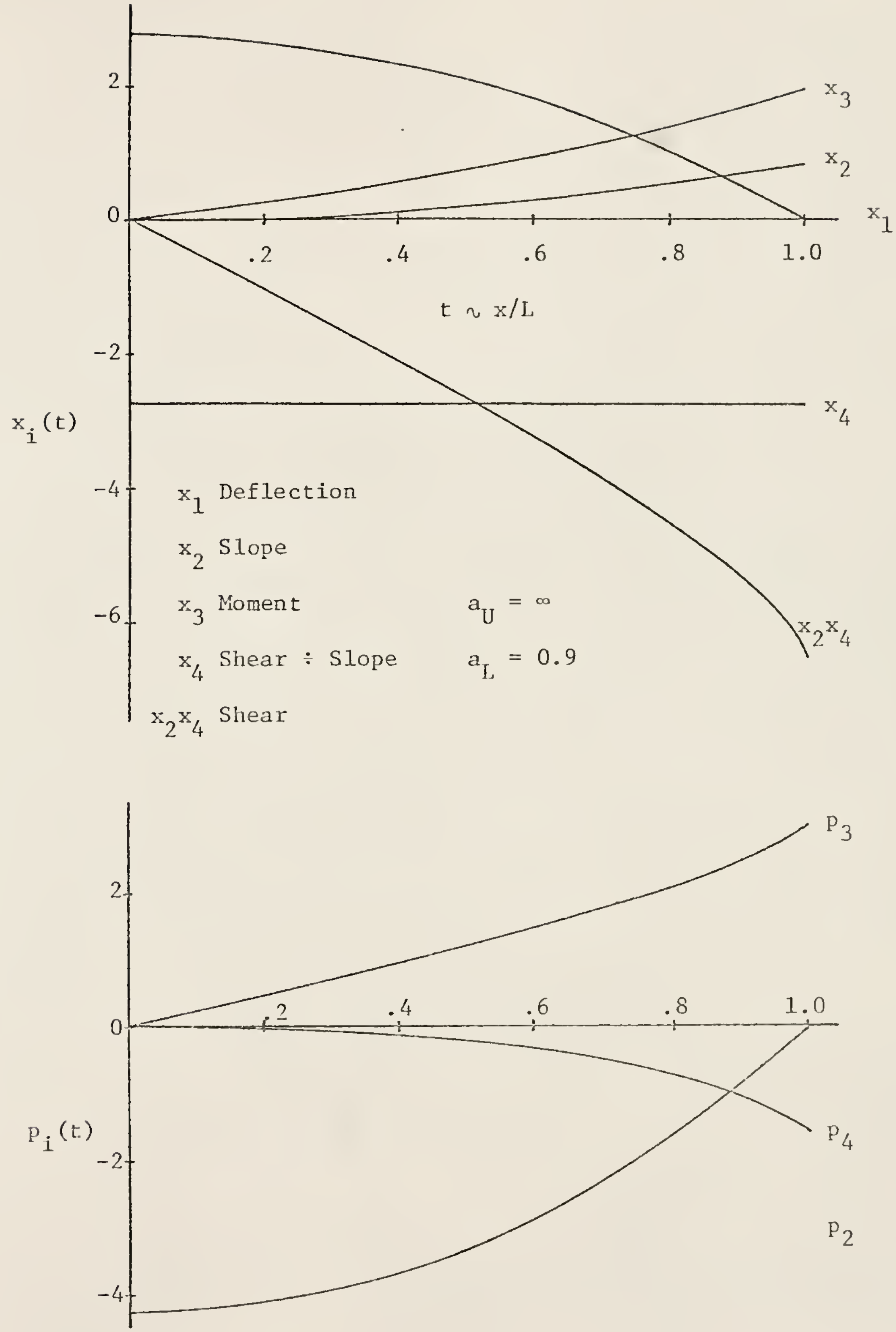

Figure 5.5 Typical State and Adjoint Variable Soltuions

Notice that for the case where density and modulus are equal to unity, the condition on specified weight simplifies to

$$J_W = \int_0^1 u_1(t)dt = 1 \tag{5.5.2}$$

where implicitly $\mu = 1$. As the value of a_L increases, it reaches an upper limit of unity: values of a_L larger than unity violate the weight condition (5.5.2). Thus, for no upper bound on u_1 there are two limiting cases associated with lower bound values,

$$a_L = 0 \qquad \text{and} \qquad a_L = 1.0$$

Values of a_L not contained in this range automatically violate some given condition. Each value of $0 \leq a_L \leq 1.0$ has a corresponding optimal eigenvalue λ_{Max} and design profile $u_1(t) \sim a(x/L)$. Several of these profiles are shown in Figure 5.6. In all but the uniform column limiting case for $a_L = 1.0$, the optimal profile has a transition region where $\underline{u}_{OPT} = u_1^*$. The point where decreasing $u_1^*(t)$ intersects the lower bound a_L occurs at a time designated t_L. Optimal control is on the lower bound for $t \geq t_L$.

A comparative measure for the effectiveness of the optimization is the ratio of λ_{Max} to the eigenvalue for a uniform column of equal weight. This latter eigenvalue is the classical Euler buckling load denoted λ_E, and corresponds to the case $a_L = 1.0$. Thus,

$$\lambda_E = \frac{P_{CR}L^2}{E_0 I_0} = \frac{\pi^2}{4}$$

The dependence of the effectiveness ratio λ_{Max}/λ_E and the lower bound intercept t_L upon a_L is shown in Figure 5.7. As is typical of simple

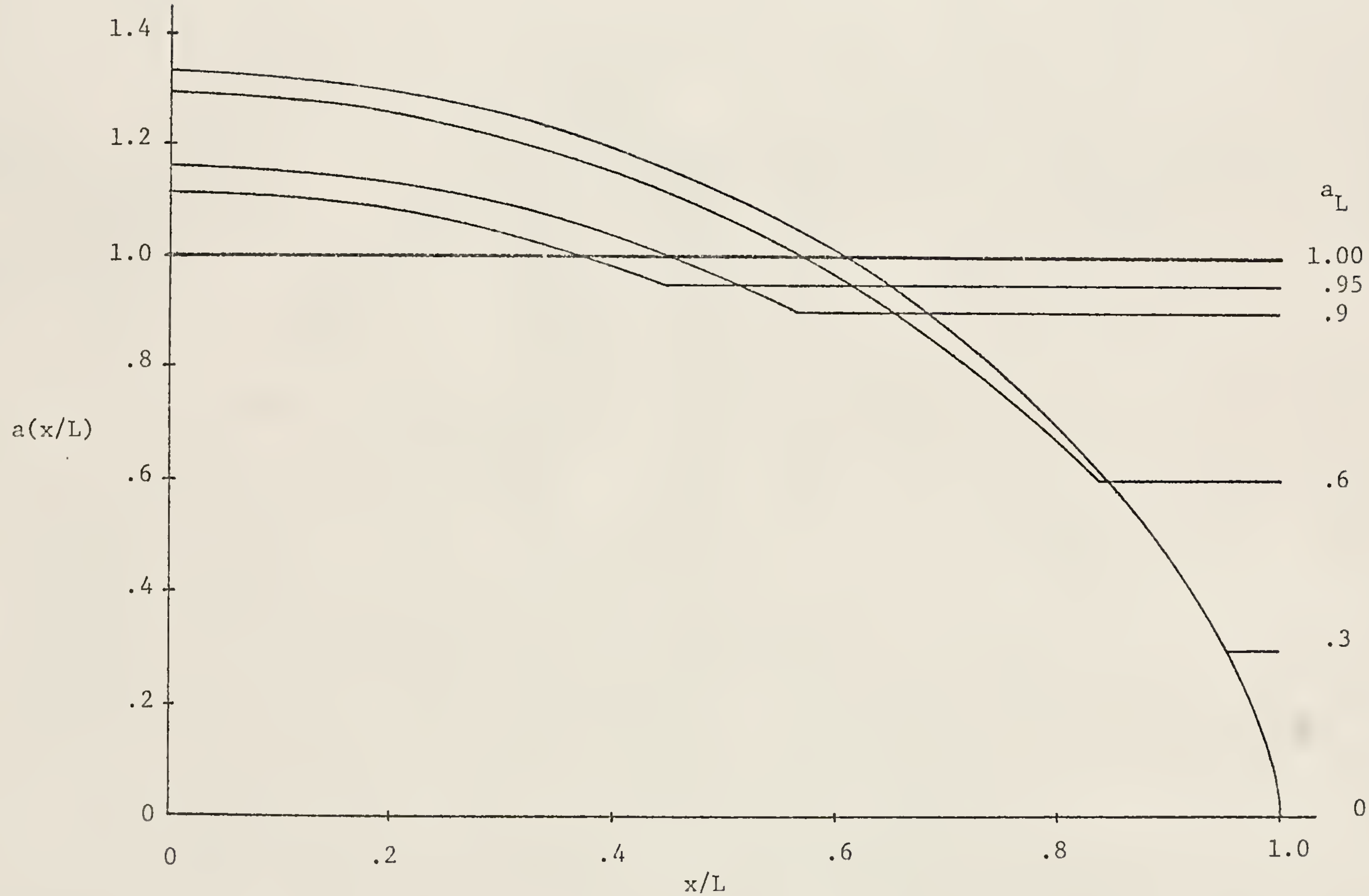

Figure 5.6 Optimal Arca Distributions Parametric in Lower Bound: No Upper Bound

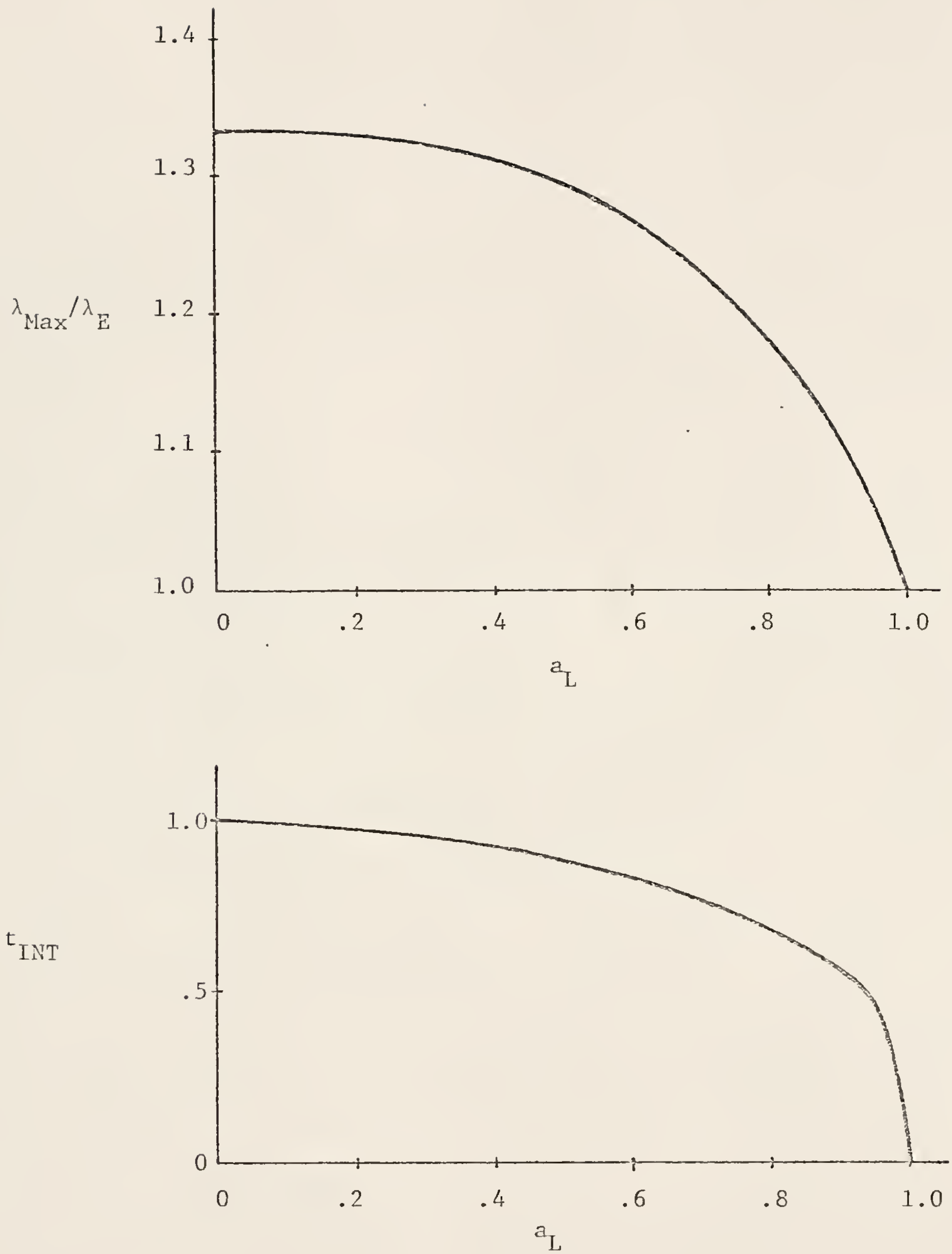

Figure 5.7 Effectiveness Ratio and Lower Bound Intercept: No Upper Bound

problems of this type, the most improvement occurs for the smallest excursions from the uniform case ($.6 \leq a_L \leq 1.0$). To allow a_L to be reduced from .6 to 0 provides much less improvement in the cost.

Observe that the maximum value for $a(x)$ occurs at $t = 0$ for $a_L = 0$, and is 1.333. Thus for $a_U > 1.333$ no optimal profile in Figure 5.6 is constrained by the upper bound on $u_1(t)$. For $a_U = 1.333$, at one point $t = 0$,

$$a(0) = a_U$$

and

$$a(t) < a_U \qquad t > 0$$

Therefore, instead of being unbounded in (5.5.1), it is only necessary that $a_U > 1.333$. An a_U this large is essentially unbounded for the given weight constraint.

Results similar to those above have been obtained for similar problems by other methods. The next examples are for a case not yet reported in the open literature. These cases could not be obtained without specifying both upper and lower control bounds. The first example has a lower bound specified with a range of various upper bounds, decreasing to the limiting case of a uniform solution. The first example of this case is

$$a_L = 0$$

$$1.0 \leq a_U \leq 1.333$$

Since the nature of the solution methods differ only in details, no solution methods are presented. Optimal profiles are presented for various values of a_U given $a_L = 0$. In Figure 5.8 the limiting cases

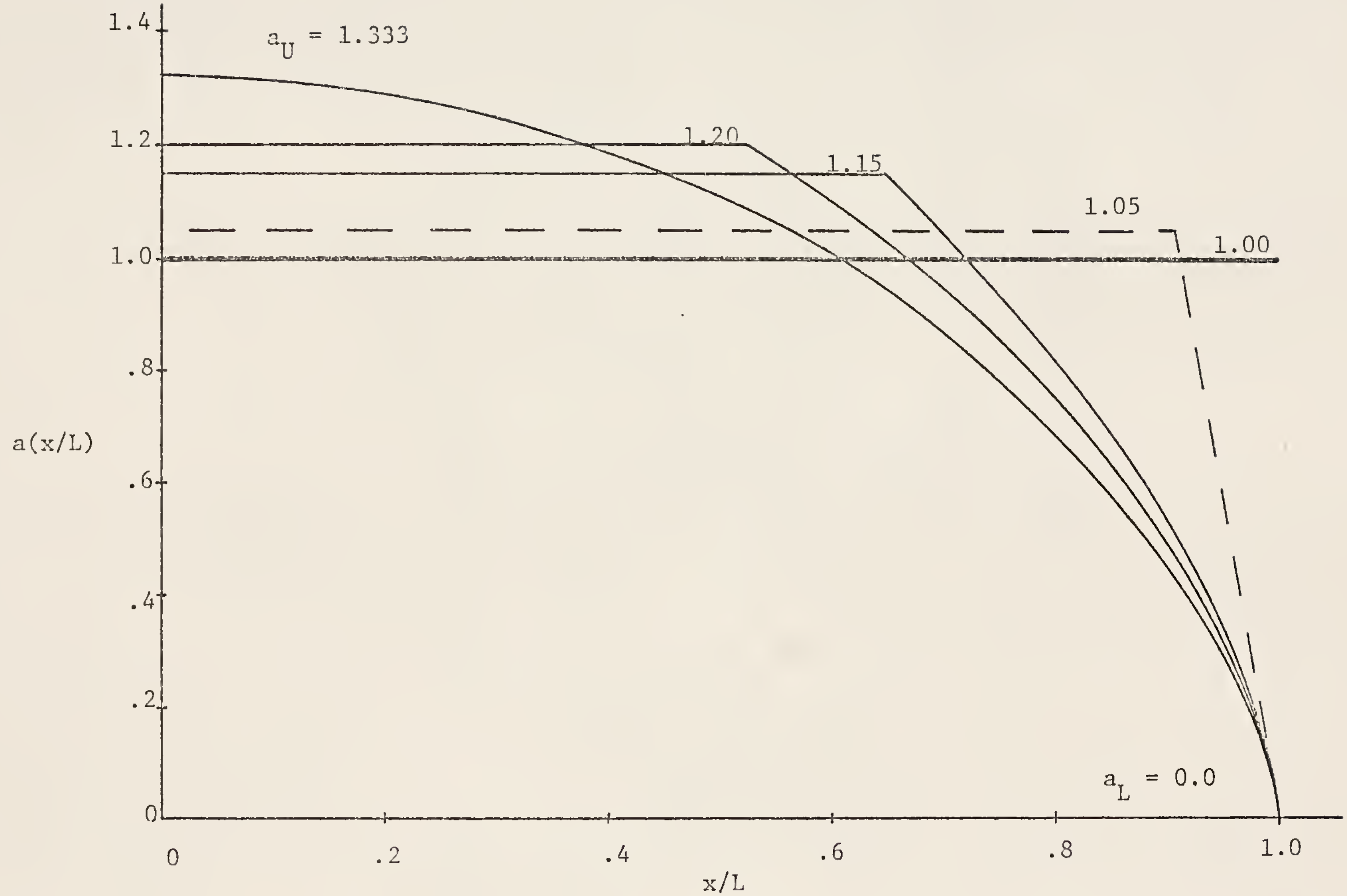

Figure 5.8 Optimal Area Distributions Parametric in Upper Bound: Lower Bound $a_L = 0.0$

are shown with three intermediate cases as well. One case is displayed as a dashed line, since it is obtained as an approximation. As $a_U \to 1.0$ the numerical procedure becomes unstable for reasons discussed in Chapter VII; the case for $a_U = 1.05$ must approach the uniform cylinder associated with $a_U = 1.0$. Because the transition region where $a(x/L) < a_U$ is approaching a vertical straight line in the limit as $a_U \to 1.0$, the approximation is taken to be:

$$a(x/L) = a_U \qquad 0 \leq x/L \leq t_U$$

$$a(x/L) = a_U \left\{1 - \frac{(t-t_U)}{(1-t_U)}\right\} \qquad t_U < x/L \leq 1$$

That value of t_U which satisfies the specified weight condition under the assumed form of the approximation is given by

$$t_U = \frac{2}{a_U} - 1$$

No further use is made of this approximation; its sole reason for existence is to provide an additional curve in Figure 5.8 to more effectively demonstrate the transition from the case with no upper bound to the uniform beam case.

The characteristic form of solutions is indicated by a plot of t_U versus a_U shown in Figure 5.9 together with the effectiveness ratio λ_{Max}/λ_E. Just as observed in the case possessing only a lower bound, when only an upper bound is present the first excursions from a uniform column ($1.0 \leq a_U \leq 1.2$) provides the greatest cost improvement.

In the preceding case, although only t_U is presented, the implication that t_L does not exist is incorrect. By observation of

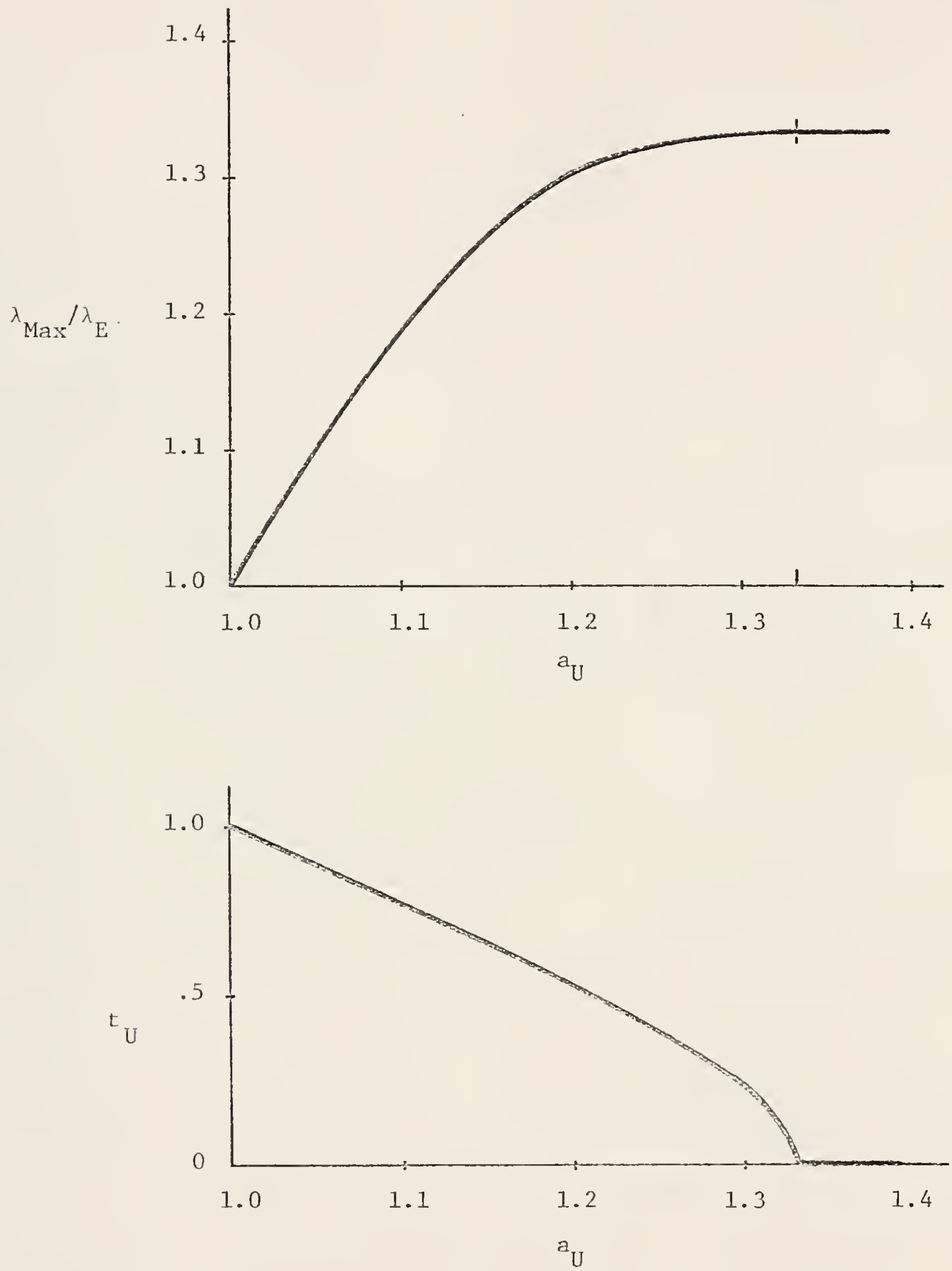

Figure 5.9 Effectiveness Ratio and Upper Bound Intercept: Lower Bound $a_L = 0.0$

Figure 5.8 and the definition of t_L it is obvious that for all cases $t_L = 1.0$. This occurs as a consequence of selecting $a_L = 0.0$; in order to demonstrate a more general behavior of the solution process, a non-zero lower bound is selected in order to provide a non-constant t_L.

The third configuration of the column buckling problem to be considered is for the control bounds

$$a_L = 0.3$$

$$1.0 \leq a_U \leq 1.327$$

where the upper limit to a_U is such that for $a_U > 1.327$ no upper bounds are ever active as u_1 is essentially unbounded from above. Optimal profiles for the limiting and three intermediate cases are shown in Figure 5.10. The dashed curve is an approximation composed of three straight line segments (numerical instability was again encountered as the uniform column limiting case was approached). Mathematically, the approximation is

$$\begin{aligned} a(t) &= a_U & & 0 \leq t \leq t_U \\ &= a_U - (a_U - a_L)\frac{(t - t_U)}{(t_L - t_U)} & & t_U \leq t \leq t_L \\ &= a_L & & t_L \leq t \leq 1 \end{aligned}$$

such that the case which satisfies the specified weight condition requires the control bound intercepts of the approximation to satisfy

$$t_U + t_L = 2\,\frac{1 - a_L}{a_U - a_L}$$

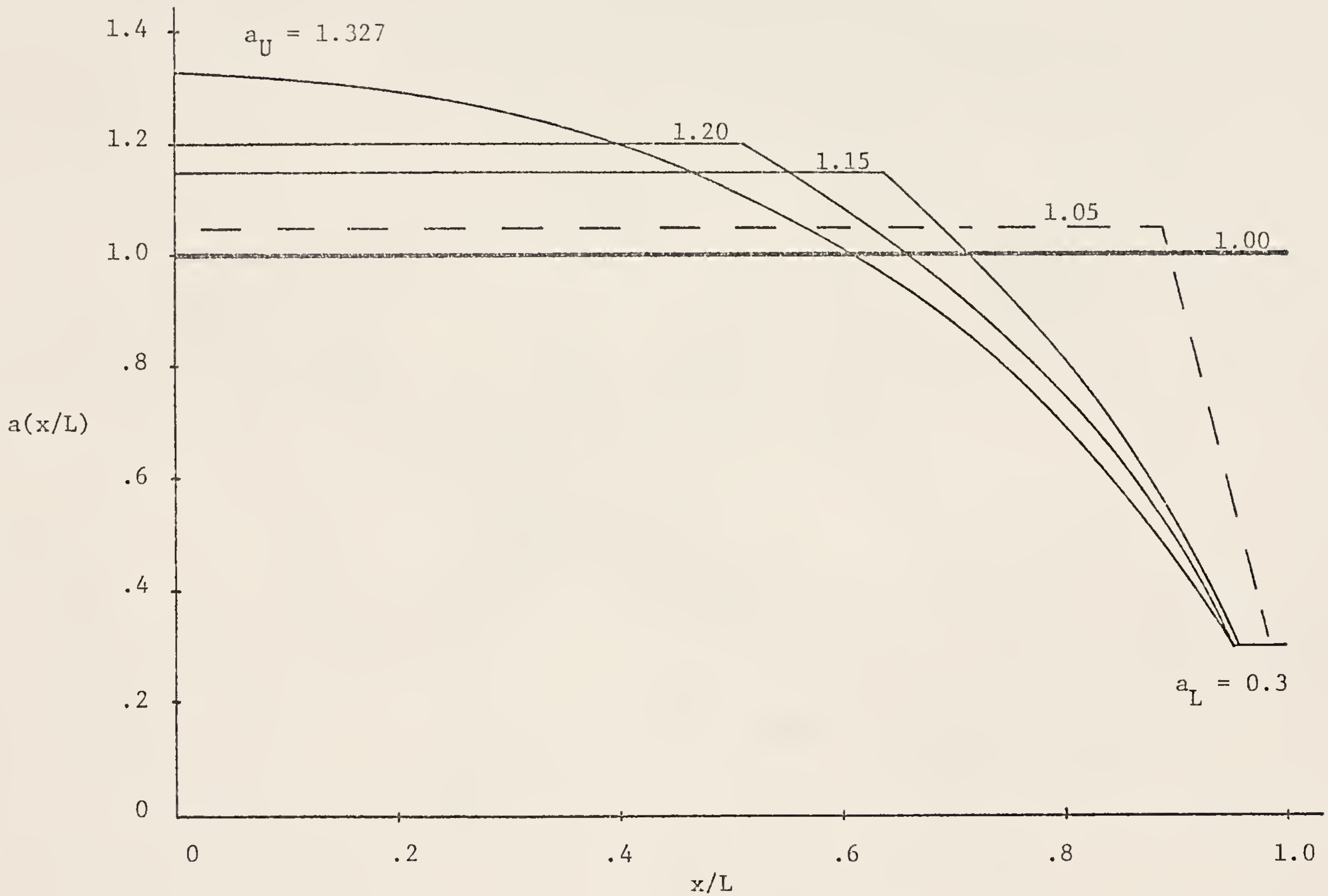

Figure 5.10 Optimal Area Distributions Parametric in Upper Bound: Lower Bound $a_L = 0.3$

From an extrapolation of intercept values associated with higher values of a_U in Figure 5.10, t_L is found to be about 0.985. The corresponding value of t_U is thus 0.882. Just as in the preceding case it must be emphasized that this approximation is included only to more clearly illustrate the transition of a(x/L) from one limiting case to the other as a_U decreases from 1.327 to 1.0.

Effectiveness of design optimization is revealed in the plot of λ_{Max}/λ_E versus a_U (see Figure 5.11). Characteristic solution forms are also depicted there in the plot of t_L and t_U dependence upon a_U. It is to be noted that t_L deviates little from the value of unity but very definitely has a functional dependence upon a_U.

5.6 Inequality Stress Constraints

Having obtained solutions for various control constraints, the next problem configuration attempted involves a mixed constraint. Initially it was decided to restrict the maximum allowable normal stress due to bending:

$$|\sigma_B(x)| \leq \sigma_{Max} \qquad\qquad 0 \leq x \leq L$$

Recognizing that for any cross section the maximum value of $\sigma_B(x)$ occurs at the "outermost fibres," the distance d(x) of these fibres from the bending axis must be determined. On the basis of assuming similar cross sections, the function f(x) appearing in the A(x) expression (5.2.1) also determines d(x), i.e.,

$$d(x) = d_0 f(x) = d_0 a^{\frac{1}{2}}(x)$$

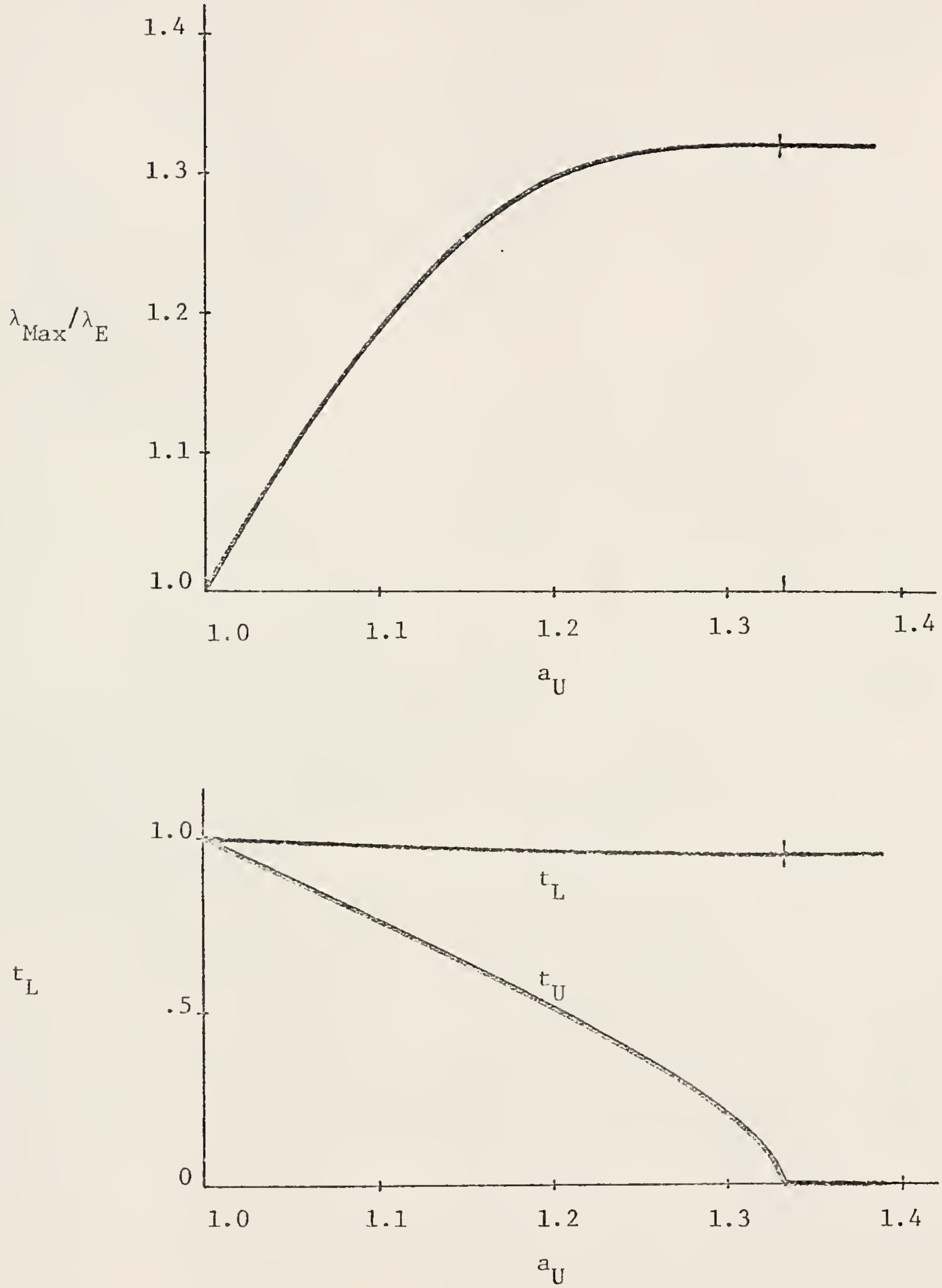

Figure 5.11 Effectiveness Ratio and Control Bound Intercepts: Lower Bound $a_L = 0.3$

where d_0 is some reference value. In terms of Bernoulli-Euler bending equation, the inequality constraint on bending normal stress is

$$|\sigma_B(x)| = \left|\frac{M_B(x)d(x)}{I_B(x)}\right| \leq \sigma\,\mathrm{Max}$$

$$= \left|\frac{(EIY'')d(x)}{I_B(x)}\right| \leq \sigma\,\mathrm{Max}$$

$$= |E(x)\ d(x)Y''| \leq \sigma\,\mathrm{Max}$$

By introducing dimensionless variables and squaring to remove the absolute value signs,

$$\left(\frac{E_0 d_0}{L}\right)^2 e^2(t)a(t)\ddot{\eta} \leq \sigma^2{}_{\mathrm{Max}}$$

which is manipulated into the desired mixed constraint form when written in terms of state and control variables:

$$x_3^2/u_1^3 - \sigma_0^2 \leq 0 \qquad (5.6.1)$$

where

$$\sigma_0 = \frac{L\ \sigma_{\mathrm{Max}}}{E_0 d_0}$$

Besides satisfying the geometric constraints, the u_1 control component must also satisfy a state dependent constraint

$$u_1(t) \geq (x_3/\sigma_0)^{2/3} \geq 0$$

Results for this case of mixed constraints are not presented. To be a valid demonstration, the optimal control ought to be constrained by the mixed constraint (5.6.1) on at least part of the interval $0 \leq t \leq 1$.

In the present example, for small σ_0 values no solution exists that can fulfill both the stress and geometric constraints. Large values of σ_0 result in all geometrically admissible control simultaneously satisfying the mixed constraint. Intermediate values produce solution iterates in the quasilinearization process that were constrained by (5.6.1) over some interval. However, the final iterate $\underline{u}_{OPT}(t)$ is not constrained by the mixed constraint at any point $0 \leq t \leq 1$. For this reason no mixed constraint examples are presented. Theoretical considerations of other possible stress constraints are given in the following paragraphs.

A more realistic stress constraint is to limit the stress component $\sigma(x)$ normal to a cross section, composed of a term $\sigma_B(x)$ due to bending and a term $\sigma_P(x)$ attributed to the end load P. With reference to the definition of θ shown in Figure 5.1, $\sigma(x)$ and its constituents are:

$$\sigma(x) = \sigma_B(x) + \sigma_P(x)$$

$$\sigma_B(x) = \frac{M_B(x)d(x)}{I_B(x)}$$

$$\sigma_P(x) = \frac{P \cos \theta(x)}{A(x)}$$

Differential calculus and trigonometry provide two identities that are used to evaluate cos (θ) as a function of x in terms of the slope of the centerline. They are

$$\frac{dY}{dx} = \tan \theta$$

$$\tan^2\theta + 1 = \sec^2\theta$$

From this,

$$\cos\theta = \left\{1 + \left(\frac{dY}{dx}\right)^2\right\}^{-\frac{1}{2}}$$

Thus, the normal stress inequality becomes

$$\sigma(x) = \frac{M_B(x)d(x)}{I_B(x)} + \frac{P}{A(x)}\left\{1 + \left(\frac{dY}{dx}\right)^2\right\}^{\frac{1}{2}} \leq \sigma_{Max}$$

By introducing dimensionless state and control variables, after much manipulation the following mixed constraint inequality is obtained:

$$\phi(\underline{x},\underline{u}) \leq 0$$

$$= -\left(u_1^{\frac{1}{2}}\right)^3 + \frac{\lambda}{\sigma_0}\left(\frac{k_0}{L}\right)^2\left(\frac{L}{d_0}\right)(1+x_2^2)^{-\frac{1}{2}}\left(u_1^{\frac{1}{2}}\right) + x_3/\sigma_0$$

Three parameters exist for this derivation,

$$\sigma_0 = \frac{L\ \sigma_{Max}}{E_0 d_0}$$

$\dfrac{k_0}{L}$ ∿ the Euler buckling slimness ratio

$\dfrac{d_0}{L}$ ∿ a geometrical slimness ratio

A simpler constraint function can be obtained by taking $\sigma_P(x)$ as merely

$$\sigma_P(x) = \frac{P}{A(x)}$$

for which

$$\phi(\underline{x},\underline{u}) = -\left(u_1^{\frac{1}{2}}\right)^3 + \frac{\lambda}{\sigma_0}\left(\frac{k_0}{L}\right)^2\left(\frac{L}{d_0}\right) u_1^{\frac{1}{2}} + x_3/\sigma_0$$

This function can be analytically solved for u_1 whenever $\phi(\underline{x},\underline{u}) = 0$. What must be solved is a cubic equation in $u_1^{\frac{1}{2}}$.

$$(u_1^{\frac{1}{2}})^3 - A(u_1^{\frac{1}{2}}) - B = 0 \qquad (5.6.2)$$

$$A = \frac{\lambda}{\sigma_0}\left(\frac{k_0}{L}\right)^2\left(\frac{L}{d_0}\right)$$

$$B = x_3/\sigma_0$$

A reduced cubic equation with real coefficients

$$y^3 + py + q = 0$$

has a discriminant

$$\Delta = -\,108R, \qquad R = (\tfrac{1}{3}\,p)^3 + (\tfrac{1}{2}\,q)^2$$

that determines the nature of the roots to the equation. For,

$\Delta < 0 \rightarrow$ one real, two complex roots

$\Delta = 0 \rightarrow$ two real, equal roots

$\Delta > 0 \rightarrow$ three distinct real roots

Consider a typical problem where the cross section is circular, and having typical parameter values:

$$k_0 = \frac{1}{2}\,d_0 \qquad \frac{d_0}{L} = 30 \qquad \sigma_0 = .06$$

it follows from

$$p = -\,\frac{1}{4}\,\frac{\lambda}{\sigma_0}\left(\frac{d_0}{L}\right)$$

$$q = -\,x_3/\sigma_0$$

and

$$\Delta = -4p^3 - 27q^3$$

$$= \frac{27}{\sigma_0^2} \left\{(\lambda^3/11.664\ \sigma_0) \times 10^{-6} - x_3^2\right\}$$

$$\simeq \frac{27}{\sigma_0^2} \left\{\lambda^3 \times 10^{-5}/7 - x_3^2\right\}$$

Thus for typical parameter values, there is a single real u_1 value whenever

$$x_3^2 > \lambda^3 \times 10^{-5}/7 \simeq 4 \times 10^{-5}$$

or

$$x_3 > (\lambda^3 \times 10^{-5}/7)^{\frac{1}{2}} \simeq 6.2 \times 10^{-3}$$

In other words, near the tip where both t and x_3 approach zero, three real distinct roots to (5.6.2) exist. Unless this possibility is excluded by a sufficiently large minimum bound to u_1, the question of which root is appropriate must be answered. Having demonstrated how complicated the analysis becomes for this particular stress constraint, no further considerations are presented. The developments are included to show that even simple stress constraints can be too complicated for ordinary methods of analysis.

CHAPTER VI

FINITE ELEMENT METHODS IN STRUCTURAL OPTIMIZATION: AN EXAMPLE

6.0 Introduction

In order to provide a contrasting comparison to the foregoing PMP examples the cantilever beam problem is solved by using finite elements in conjunction with mathematical programming techniques. It is assumed the beam has a constant specified width in order to better exemplify the concepts contained in this chapter. Numerical solutions obtained by a feasible directions algorithm are compared to the PMP results. This example serves to illustrate the principles used in the theoretical derivations.

6.1 Finite Element Problem Statement

The same cantilever beam problem solved in Chapter IV is treated here with different techniques. In what follows the structural system is modeled by and solved with finite elements. A direct consequence of this mathematical representation is that the optimization problem is transferred from a vector space of continuous functions to a space of finite dimension. Since no contributions to finite element theory are made, the method itself is not discussed. The reader is referred to Zienkiewicz (1971) for details of the method. Bernoulli-Euler Bending Theory is applied to a fixed length cantilever beam of constant width

and material properties, which is symmetric about its central axis. The problem is to find the variable height h(x) that gives the minimum tip deflection due to its own height, subject to hard constraints of maximum and minimum allowable heights.

A higher ordered element is used to obtain better accuracy; both the displacement and its derivative at the nodes are prescribed to be the independent variables. For this simple problem, n nodes divide the beam into (n-1) elements of equal length.

Hence, the displacement and its derivative are the deflection and rotation of the beam's centerline at the locations of the nodes. It is also assumed that each element is linearly tapered, whereby the Bernoulli-Euler bending equation for element "i" takes the form

$$EI(x)\,\frac{d^2y}{dx^2} = M(x)$$

$$I(x) = \frac{1}{12}\,wh^3(x)$$

Position along the beam x, height h(x), and displacement of the beam y(x), and slope $\theta(x)$ of the centerline, are approximated by vectors composed of discrete elements:

$$\underline{x} = [x_1\ x_2\ \dots\ x_n]^T$$

$$\underline{h}(\underline{x}) = [h(x_1)\ h(x_2)\ \dots\ h(x_n)]^T$$

$$\underline{y}(\underline{x}) = [y(x_1)\ y(x_2)\ \dots\ y(x_n)]^T$$

$$\underline{\theta}(\underline{x}) = [\theta(x_1)\ \theta(x_2)\ \dots\ \theta(x_n)]^T$$

Whenever a height distribution is specified, a finite element solution for the beam's slope and deflection can be obtained.

6.2 Mathematical Programming: Gradient Projection Method

With the structural analysis given in the form of a finite element solution, the mathematical statement of the optimization problem is

Given: (i) material properties ρ, E

(ii) dimensions L, w

(iii) number of nodes n

Find: $\underset{\underline{h}(\underline{x})}{\mathrm{Min}} \; [y(x_n)]$

subject to $b \leq h(x_i) \leq d \qquad i = 1,\ldots,n$

This may be stated as a nonlinear mathematical programming problem, whose optimality conditions are prescribed by the Kuhn-Tucker Theorem given in section 3.2. For this particular beam problem corresponding quantities of the theorem are

$$\underline{x} \sim \underline{h}$$

$$F(\underline{x}) \sim \underline{y}(\underline{x})\Big|_{x=L}$$

$$g_j(\underline{x}) \sim \begin{cases} h(x_i) - d & i = 1,\ldots,n \qquad j = i \\ b - h(x_i) & i = 1,\ldots,n \qquad j = i+n \end{cases}$$

The numerical algorithm used to obtain the optimal solution satisfying the Kuhn-Tucker conditions is a modified gradient projection method described by Kowalik (1970). Briefly speaking, it uses projections of the cost function gradient into a subspace satisfying currently active constraints. Starting from a feasible design point $\underline{h}_0$, a sequence of feasible designs is generated such that

$$\underline{h}_0, \underline{h}_1, \underline{h}_2, \ldots, \underline{h}_q, \underline{h}_{q+1}, \ldots, \underline{h}_{OPTIMAL}$$

where

$$\underline{h}_{q+1} = \underline{h}_q + \alpha_q \hat{S}$$

$$\hat{S} = - P\nabla F$$

$$\alpha_q > 0$$

$$P = I - N(N^T N)^{-1} N^T$$

A derivation of projection matrix P is found in Kowalik (1970). Columns of N are the gradient vectors of the active constraints; thus, N is an $(n \times r)$ matrix where r is the number of active constraints and $0 \leq r \leq n$. If $r = 0$, then by definition N is a null matrix and P reduces reduces to the identity matrix I. With P so defined, those components of $-\nabla F$ that lead to a constraint violation are subtracted from the direction of steepest descent. In this manner $\hat{S}$ is the direction which best improves the cost function without leaving the region of feasible designs where $g_j(\underline{x}) \leq 0$. From the form of $g_j(\underline{x})$ in this particular problem, the typical element N_{ik} of N is defined as:

$$N_{ik} = \frac{\partial g_{\alpha_k}}{\partial x_i} \qquad i = 1,\ldots,n \qquad k = 1,\ldots,r$$

where α_k are those indices $1 \leq j \leq 2n$ associated with the r active constraints for which $g_j(\underline{x}) = 0$. Hence,

$$N_{ik} = 0 \qquad i \neq \alpha_k$$

$$N_{ik} = \begin{cases} 1 & i = \alpha_k \quad \text{and} \quad \alpha_k \leq n \\ \\ -1 & i = \alpha_k \quad \text{and} \quad \alpha_k > n \end{cases}$$

There is no difficulty in calculating the gradients required by N; since the $g_j(\underline{x})$ are specified functions the gradients can be determined explicitly. Derivatives required to evaluate ∇F are not so easily obtained. To circumvent the lack of an explicit formulation, the derivatives are obtained as follows. The i^{th} component of ∇F is found numerically by

$$\frac{\partial F}{\partial x_i} = \frac{F(\underline{x} + \Delta\underline{x}) - F(\underline{x})}{||\Delta\underline{x}||}$$

where the only nonzero component of $\Delta\underline{x}$ is

$$\Delta x_i = \frac{1}{10} \; x_i$$

To avoid instability as the solution is approached, the increment is modified to

$$\Delta x_i = \frac{1}{10} kx_i$$

where k is the maximum percent change of all components of $\underline{x}$ from the previous iteration. No numerical instabilities were encountered with this scheme; however, if too small a value of numerical tolerance is chosen, the convergence criteria are never satisfied and the algorithm begins to "hunt" in the neighborhood of the optimal solution. A simplified flowchart is shown in Figure 6.1.

To furnish a comparison standard, the Bernoulli-Euler equation for a beam with a linearly tapered center section was integrated analytically. This revealed some interesting qualitative results which are discussed below, with quantitative results given in the following section. First, the expression for the second derivative of y(x) is

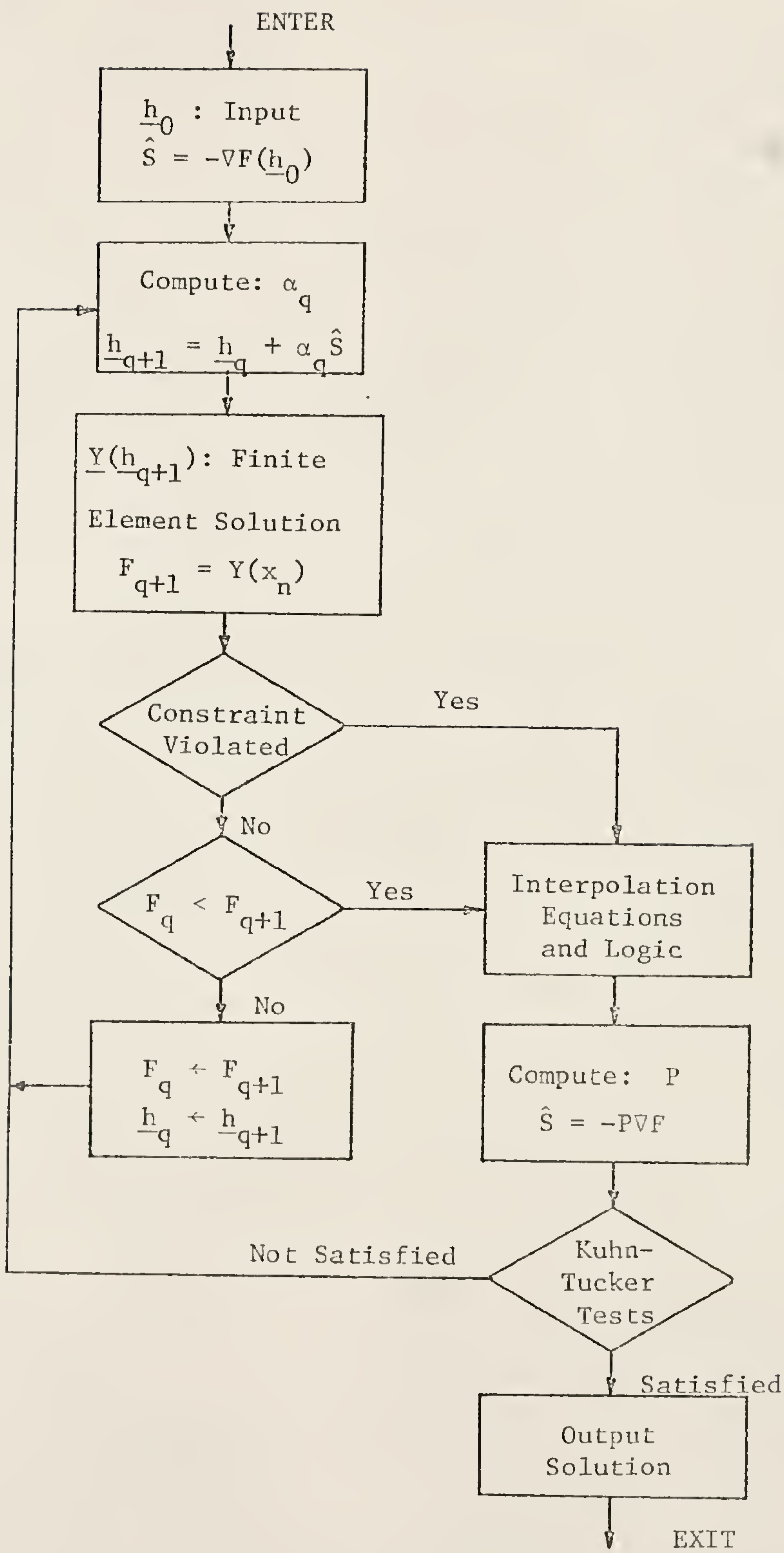

Figure 6.1 Simplified Flowchart of Algorithm

only piecewise smooth due to $I(x)$. To integrate the equations for the linearly tapered beam required dividing the range of x into three reqions. In the region $0 \leq x \leq x_1$ the height $h(x) = d$ and $y(x)$ is a quartic polynomial in x. Variable height complicates the equation in the region $x_1 \leq x \leq x_2$; however, the centerline deflection is still a function of a single variable, $u(x)$, defined as

$$u(x) = 1 - \left(\frac{x}{L} - \frac{x_1}{L}\right) A$$

$$A = 2\,\frac{L}{d}\,\tan\alpha, \text{ a constant}$$

where

$$\tan\alpha = \frac{1}{2}\left(\frac{L}{d}\right)^{-1} \frac{(1 - \frac{b}{d})}{\left(\frac{x_2}{L} - \frac{x_1}{L}\right)}$$

such that

$$u(x_2) = \frac{b}{d}$$

One of the complicated terms of the expression for $y(x)$ in this region of the beam is

$$[2\ \ln\,(u(x)) - u(x)]$$

which is undefined at $x = x_2$ when $b/d = 0$. Within the third region $x_2 \leq x \leq L$, the height $h(x) = b$ and the centerline deflection is a quartic polynomial in $(x - x_2)$.

It is shown in Hornbuckle et al. (1974) that in all three regions the deflection could be written in nondimensionalized form

$$\frac{y(x)}{y_0} = \text{f c t } (x, \frac{b}{d})$$

$$y_0 = 12 \frac{\rho g L^2}{E} \left(\frac{L}{d}\right)^2 \tag{6.2.1}$$

This result supports Dixon's statement that d/L is not a parameter of the dimensionless system. These equations indicate that dimensionless results indeed depend only upon b/d. However, for a particular beam of given dimensions, the actual tip deflection does increase as the square of the slimness ratio L/d.

6.3 Results

Before any attempts were made to obtain results, certain basic questions had to be answered. First, the integration inherent to the finite element solution as described in Zienkiewicz (1971) is done numerically, thereby requiring determination of the proper step size. It was found that with the range of parameters used in this problem, ten integration intervals per element gave the desired accuracy without excessive computation. Secondly, there was a question of how many nodes are required to give a valid representation of the beam. Because the system is so simple, as few as two finite elements produce good structural results vis-à-vis tip deflection. However, unless eight or sixteen elements were used, the approximation of h(x) by $\underline{h}(\underline{x})$ was not acceptable. The results for eight or sixteen elements were almost identical. (For the latter two cases the execution times on an IBM-370/165 were 15 seconds and 110 seconds, respectively.)

Basic operation of the algorithm and convergence to the optimal solution is illustrated in Figure 6.2 for a typical value of b/d. The profile is displayed at each stage in the design iteration process, including the final design. A "stage" in the design process occurs whenever another constraint is encountered, requiring the recomputation of ∇F, P, and $\hat{S}$. To simplify the figure only nine nodes were used; the optimal profile for the beam having linear taper is shown for comparison. The listing below indicates how constraints are encountered as the numerical process approaches the optimal design. Tip deflections for each design iterate are included. Thus, for the design process depicted by Figure 6.2,

No. of Active Constraints	Tip Deflection		
0	.108879	...	Initial Design
1	.027556		
2	.020673		
3	.017103		
4	.016646		
5	.015351	...	Final Design
Comparison	.015332	...	Linear Taper Design

In the earlier studies, the problem was found to have a single parameter b/d; however, two different beam profiles h(x) were obtained for this problem by various authors. It was found with the finite element method that for numerical precision corresponding to slide rule accuracy the optimal profile is an almost linearly tapered beam. Greater numerical accuracy provides a distinct curvature in the tapered portion of the beam. This is shown in Figure 6.3 by the two curves generated with the finite element method. The profile depicted by the

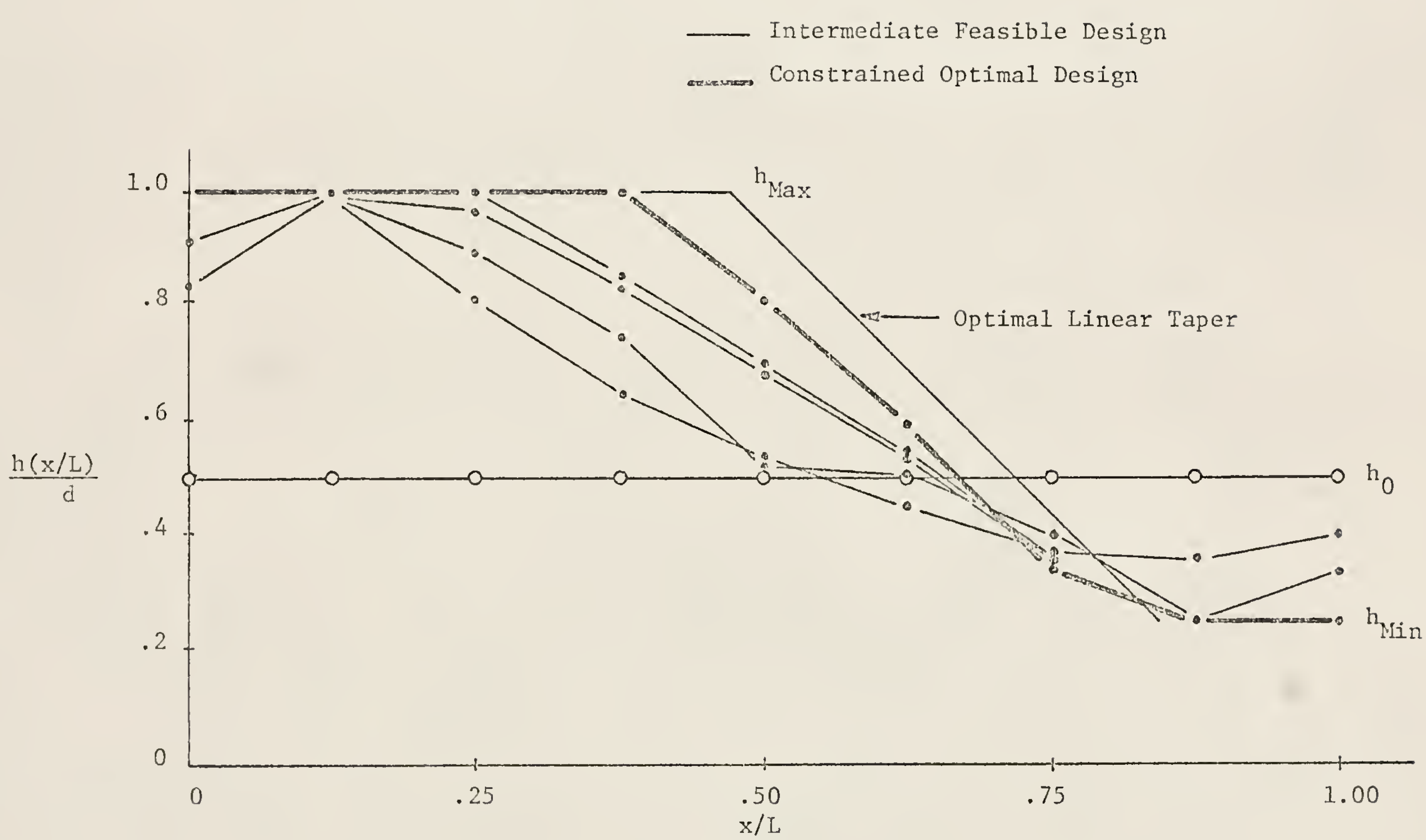

Figure 6.2 Design Iteration to Optimal Profile

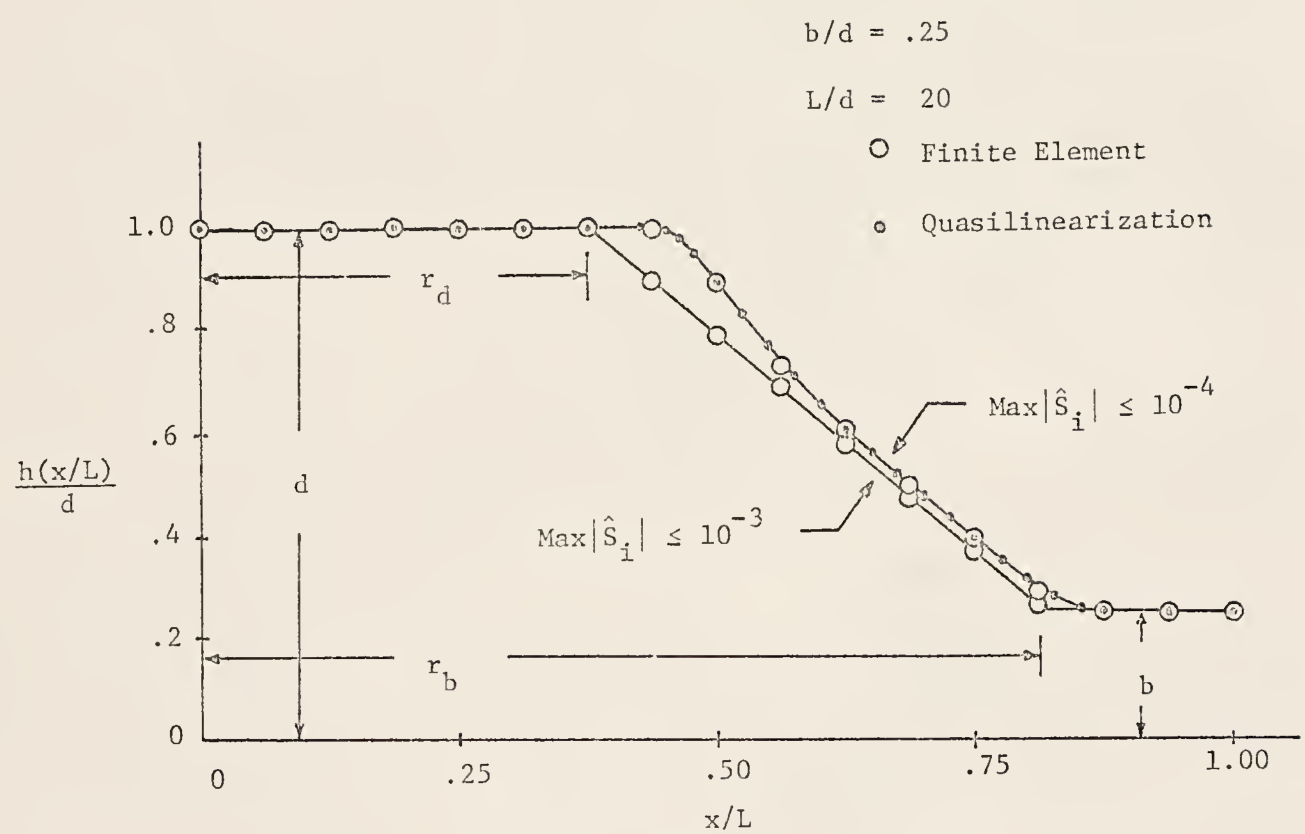

Figure 6.3 Optimal Profile: Comparison of Methods and Numerical Tolerance

dots is taken from the work by Boykin and Sierakowski (1972), and is presented for corroboration. For this latter curve the absolute magnitude of the error in satisfying the differential equations is less than 10^{-15}.

Since tip deflection is the cost function, the tip deflection from the finite element algorithm was compared to that from the linearly tapered beam (Figure 6.4). Differences were less than .5% except near b/d = 0. This is attributed to the natural logarithm term in the linear taper solution discussed previously. No computer results were obtained for the linear taper solution with b/d almost zero because the natural logarithm term became excessively large. This may explain why all earlier studies using this example problem contain no results for $b/d \leq 0.1$. However, the finite element algorithm converged to a solution for b/d = 0 with no difficulty.

Further analysis centered on a statement in which Bellamy and West (1969), claim: ". . . as b/d increases, the midsection of the beam profile reduces in length and increases in slope." Having found quite close agreement with the linear tapered tip deflection, we expected at least qualitative agreement for the angle of taper α. Angle of taper for the linearly tapered beam and the finite element results are shown in Figure 6.5. Notice that both curves approach zero in the limit as b/d approaches one. Of greater importance is the fact that the angle of slope for both methods is reasonably close only for $b/d \leq 0.5$. Figure 6.6 shows that for $b/d > 0.5$, if the beam is divided into sixteen finite elements, the tapered section contains only three

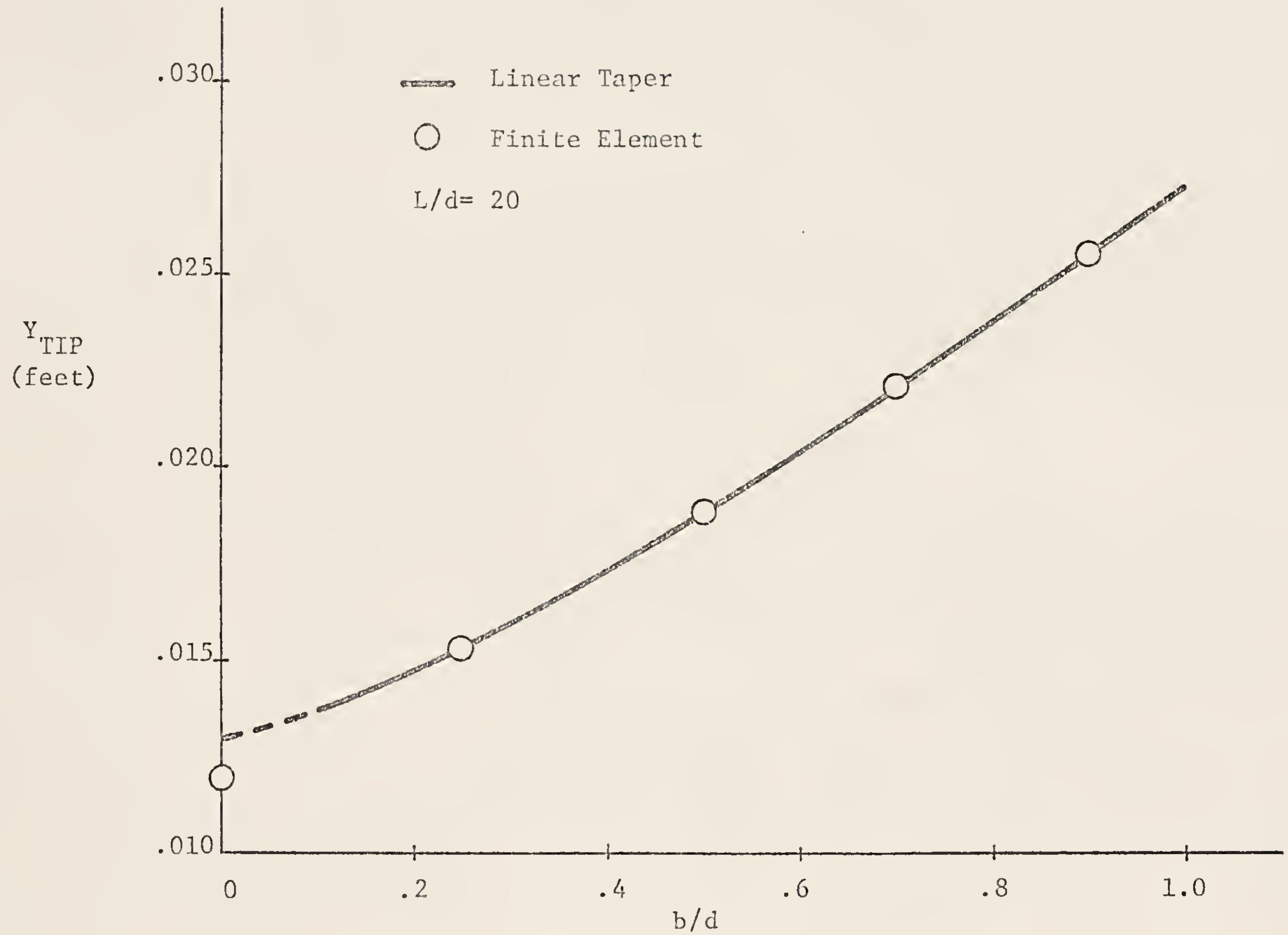

Figure 6.4 Tip Deflection Versus b/d

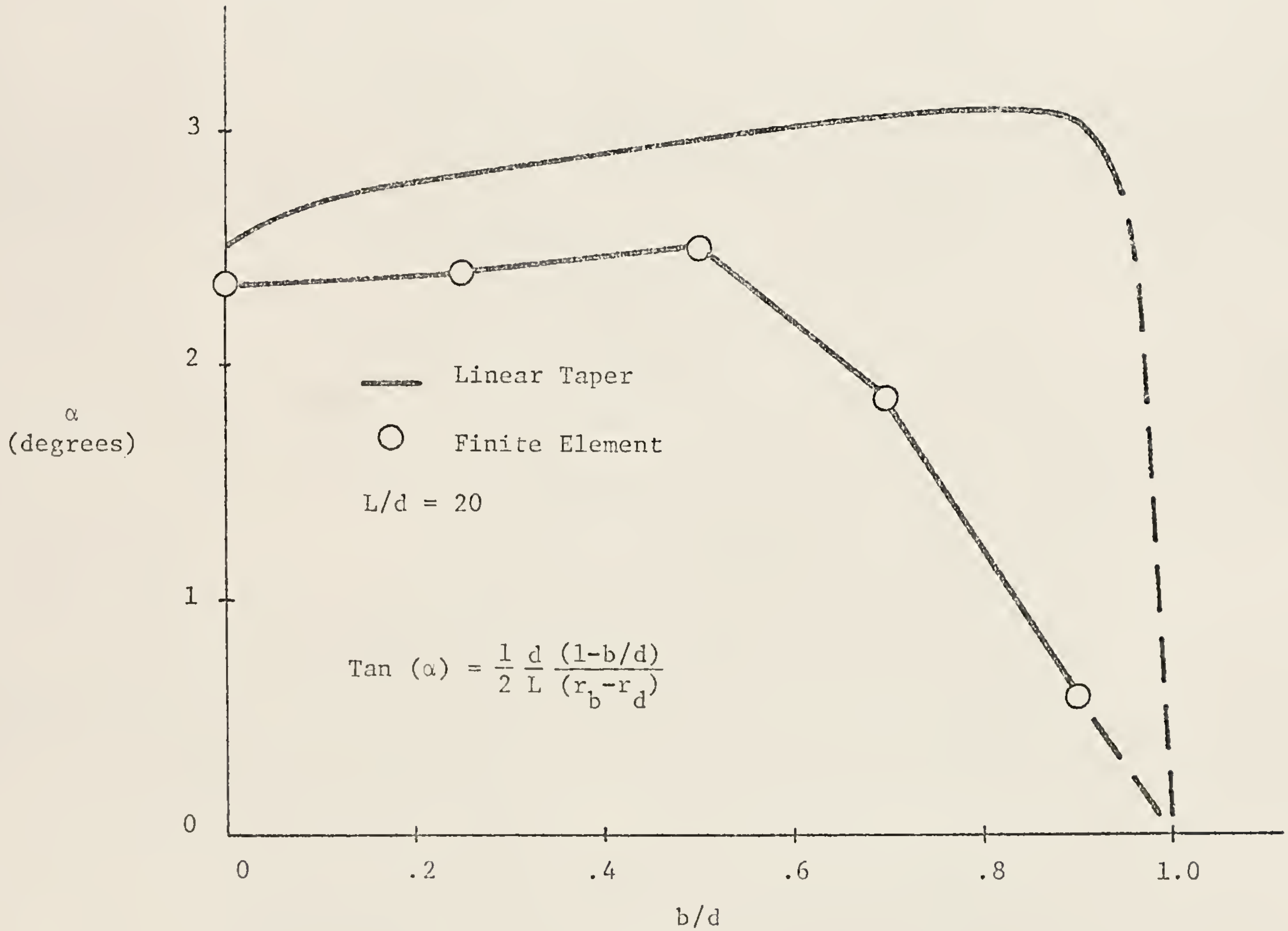

Figure 6.5 Angle of Taper Versus b/d

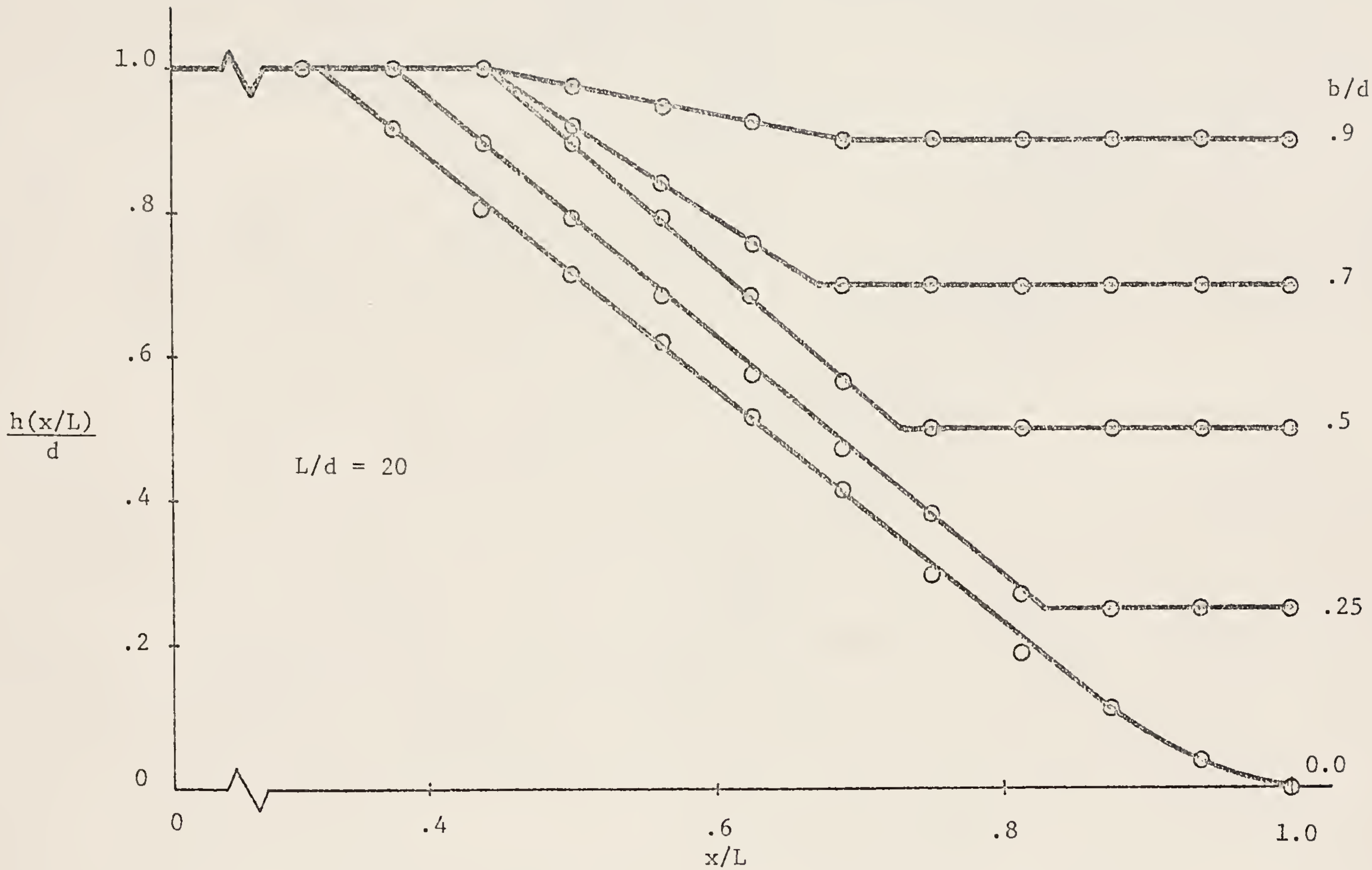

Figure 6.6 Optimal Beam Profiles via Finite Elements

nodes because it spans such a short section of beam. This is believed to be insufficient for accurate representation. The same shortcoming is also apparent in Figure 6.7, which shows the optimal profile dimensions reported in the various studies. For $b/d > 0.5$, the value of r_d obtained by the finite element method diverges from the other solution method results.

To verify independence of the solution to L/d, a series of runs were made with fixed b/d and the slimness ratio L/d ranging from five to eighty. Resulting profiles at first suggested that L/d is a parameter, but closer analysis revealed there was an oversight in the use of numerical tolerances. Initially a single tolerance was applied to both geometrical constraints and the Kuhn-Tucker test. The former is independent of relative beam shape, but the latter is not; this is implied by the form of the nondimensionalization coefficient given by (6.2.1).

Basically, the projection matrix P deletes those components from $-\nabla F$ which are normal to an active geometrical constraint boundary. Since all numerical schemes contain inherent accuracy limits, the Kuhn-Tucker test is considered to be satisfied when at those points not on a boundary

$$\left| - \frac{\partial y(x_n)}{\partial x_i} \right| < \varepsilon_G$$

Yet as the slimness ratio L/d increases, for any point on the beam the magnitude of the component of the gradient also increases. By requiring the Kuhn-Tucker test to satisfy the same tolerance for all values of L/d, as L/d increases the solution is effectively forced to become more accurate. In fact, more accuracy than is possible from the

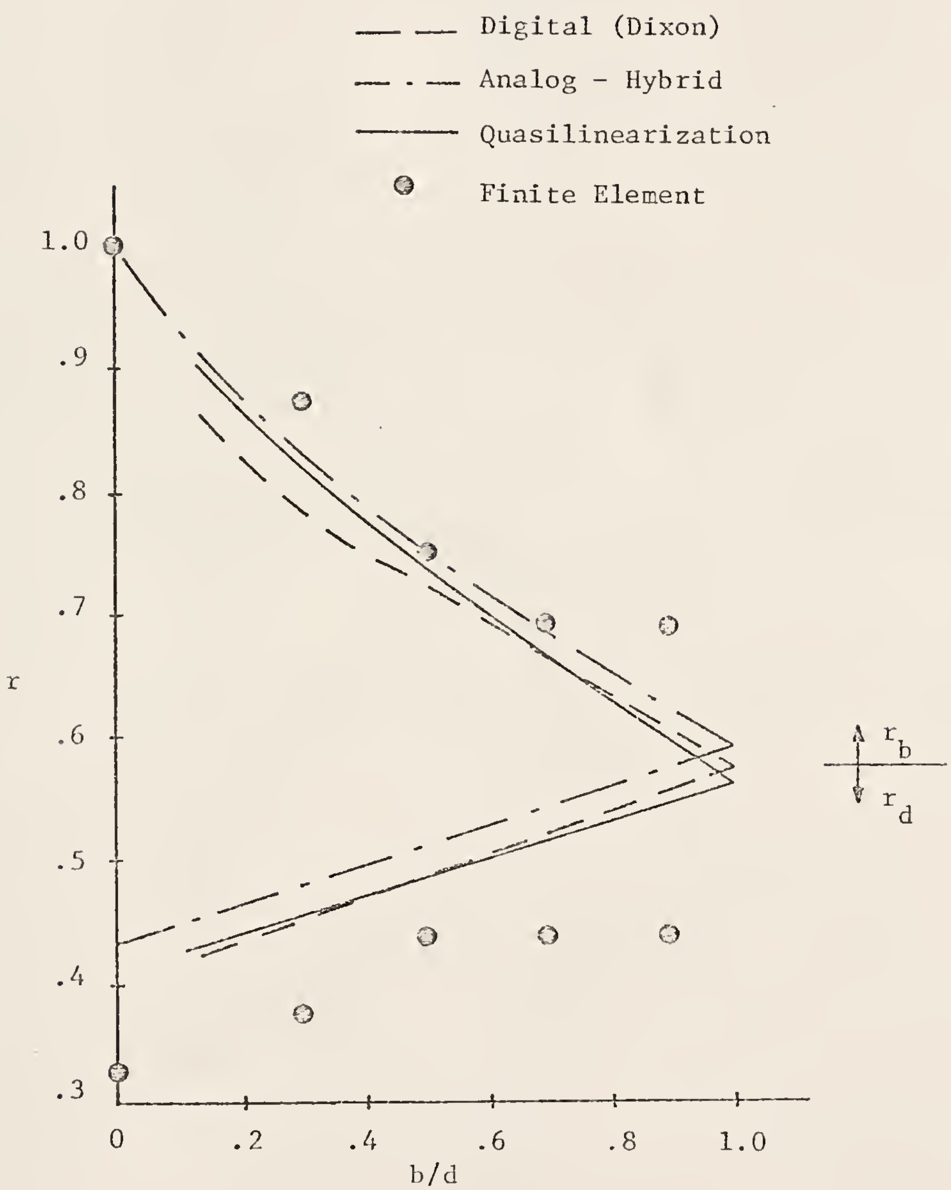

Figure 6.7 Dependence of Intercept Location upon b/d

algorithm was required and the system began hunting around the solution, failing to converge. By numerical experimentation and curve fitting it was found--Hornbuckle et al. (1974)--that uniform accuracy could be obtained for this particular problem when the tolerance used in the Kuhn-Tucker test is

$$\varepsilon_G = \varepsilon_0 + k_0(L - L_0)^{N_0}$$

$$\frac{L_0}{d} = 5$$

$$N_0 = 2.77825$$

$$k_0 = 1.184045 \times 10^{-6}$$

Subscript zero indicates values associated with reference length corresponding to L_0/d; the value of five was chosen because it represents an approximate lower bound for the validity of Bernoulli-Euler bending theory. In uniform beams whenever $L/d < 5$, the effect of shear is no longer negligible. As an example of why care should be exercised in selecting the range of parameters to generate data, Dixon (1967, 1968) obtained results for $0.2 \leq b/d < 4$; having assumed the effect of shear negligible, he considered only the range of L/d where it is significant.

CHAPTER VII

COMMENTS ON NUMERICAL INSTABILITY IN THE QUASILINEARIZATION ALGORITHM

7.0 Introduction

Quasilinearization is an indirect type of numerical scheme in which a solution is obtained for the differential equations that must be satisfied by the optimal solution. Since a standard IBM SHARE program is used without modification, the method is not discussed. Interested readers are referred to any standard textbook on numerical techniques, or to Ghosh (1973).

This type of algorithm requires an initial guess of the state and then iterates from that guess to the solution. Control functions associated with the final iterate are the optimal control functions actually sought. If the process converges at all, the convergence is rapid and typically requires fewer than ten iterates. However, the paramount difficulty of using this method is to select an initial guess which is "good" enough to yield convergence. This chapter contains some numerical data related to specific cases of the example problems where convergence difficulties are encountered.

7.1 Computer Program Convergence Features

As explained in section 4.4, convergence is measured by whether or not the iterate satisfies the differential equation. In terms of

the quantity ERROR defined by (4.4.2), in subroutine QUAS1 the test for numerical instability involves three program parameters:

E1MAX	the value of ERROR for the initial guess
E2MAX	the value of ERROR for some subsequent iteration
BLO	a specified instability parameter

Whenever the ratio of E2MAX to E1MAX exceeds BLO the program terminates on an error exit. Typical values of BLO are 10 to 20. The primary mode of convergence is for E2MAX, which decreases with each iteration under normal circumstances, to become smaller than a specified value (assumed to be 0.5×10^{-6}). Other special modes of convergence contained in QUAS1 were never used by any of the cases run for either example problem.

The purpose of this test is twofold. In some instances the initial guess is not "good" enough and the first few iterations possess E2MAX values greater than E1MAX, but is sufficiently good for the algorithm to ultimately converge. The other purpose is to prevent excessive computation for those cases where the initial guess is so bad that the numerical process is divergent.

Another related convergence test is required by the subroutine RECIP which performs required matrix inversion. This test involves a program parameter SUM which is defined in terms of a matrix $\underline{C}$ to be inverted, and the inverse $\underline{C}^{-1}$ obtained numerically by RECIP. If C_{kj} and C^{-1}_{ik} denote appropriate elements respectively, then

$$\mathrm{SUM} = \sum_{k} \sum_{i \neq j} |C^{-1}_{ik} \, C_{kj}|$$

If the matrix has been inverted exactly the value of SUM is zero. Whenever SUM is less than program parameter OFFDG the inversion is considered to be sufficiently accurate. The value selected for all cases for both example problems is 5×10^{-15}. Whenever the inverted matrix fails to satisfy the test on SUM, the subprogram iterates upon the inverse to improve it. If the improved inverse satisfies the accuracy requirement, the result is returned to QUAS1. After five unsuccessful improvement attempts, results of the fifth attempt are returned to QUAS1 A singular matrix results in immediate termination of all calculations via an error exit.

During the operation of the quasilinearization algorithm two different matrices must be inverted. One of these, $\underline{D}_i$, exists for every element i and each separate $\underline{D}_i$ must be inverted. From all of the $\underline{D}_i$ another matrix $\underline{C}$ is defined which also requires inversion. The specific form of these matrices is not important; however, their definitions may be found in Ghosh (1973, pp. 52-58). It is noteworthy that in the cases run for the example problems no difficulties were experienced in the inversion of any $\underline{D}_i$ matrix. All matrix inversion problems were associated with the $\underline{C}$ matrix.

7.2 Numerical Instabilities for Cantilever Beam Example

An initial guess for the beam problem was obtained from the solution for a uniform beam, that is, for parameter values a/c = b/d = 1. Cases were then run for a range on both parameters from 0.1 to 1. As would be expected, no convergence difficulties were encountered with

combinations of parameter values corresponding to a nearly uniform beam. In fact, with the initial guess based upon the uniform beam, only three cases failed to converge. For this initial guess given by (4.4.3) the following three cases exhibited numerical instability.

$$\begin{aligned} a/c = 0.1 \qquad b/d &= 0.3 \\ &= 0.2 \\ &= 0.1 \end{aligned} \tag{7.2.1}$$

All other combinations of parameter values resulted in convergence by the quasilinearization algorithm.

Since all of those cases which did converge showed very little change in structural state components $x_i(t)$, $i = 1,\ldots,4$, the uniform beam initial guess was replaced by a guess based upon the case where $a/c = 0.1$ and $b/d = 0.4$. It was suspected that the adjoint variable guesses were not sufficiently "good." Hence, the equations (4.4.3) were replaced by:

$$\begin{aligned} x_1(t) &= .025 * t^2 \\ x_2(t) &= .040 * t \\ x_3(t) &= .100 * (1-t) \\ x_4(t) &= -.400 * (1-t)^2 \\ p_3(t) &= 5.00 * t^3 \\ p_4(t) &= -1.25 * t^3 \end{aligned} \qquad 0 \le t \le 1 \tag{7.2.2}$$

With this initial guess only one case corresponding to the three sets of parameter values (7.2.1) failed due to numerical instability; the first two cases of these cases converged.

Having succeeded in forcing two additional cases to converge by improving the initial guess, this same approach was tried with the third case as well. The initial guess was based upon the converged solution for the case with $a/c = 0.1$ and $b/d = 0.2$. Equations (7.2.2) were replaced by

$$\begin{aligned} x_1(t) &= .020 * t^2 \\ x_2(t) &= .040 * t \\ x_3(t) &= .080 * (1-t)^3 \\ x_4(t) &= -.350 * (1-t)^3 \qquad 0 \leq t \leq 1 \\ p_3(t) &= 13.0 \quad * t^4 \\ p_4(t) &= -2.00 \quad * t^4 \end{aligned} \tag{7.2.3}$$

With this initial guess the case for $a/c = .1$ and $b/d = .1$ converged also. This last case gave a complete data set for parameter values

$$.1 \leq a/c \leq 1$$

$$.1 \leq b/d \leq 1$$

in increments of .1. No cases for parameter values equal to zero were run since the parameters appear in the denominator of various expressions.

As mentioned earlier, indirect methods such as quasilinearization generally converge rapidly if the initial guess is sufficiently good. To try to determine why the three cases failed to converge, all of the cases were closely examined. Two characteristics were observed:

(i) There is little change in the maximum value of each separate state component, regardless of the parameter values. Curvature of $x_3(t)$ and $x_4(t)$ did change for lower parameter values;

this is evident in different expressions used for the initial guess of these two variables. However, the variable that actually caused the instability termination was $p_3(t)$ and not a state component.

(ii) The maximum value of both adjoint variables increases with decreasing parameter values, with an associated increase in curvature.

With $p_3(t)$ isolated as the source of convergence difficulties for cases with small parameter values, $p_3(t)$ was plotted for several combinations of small parameter values.

The increased curvature mentioned above is shown in Figure 7.1. All cases that failed to converge exhibit a "numerical instability termination" for which $p_3(t)$ becomes divergent. Maximum error in satisfying the $\dot{p}_3$ equation occurs in the region $.755 \leq t \leq .965$ for these divergent runs. From the figure it can be seen that this is the region of greatest change in the curvature of the solution $p_3(t)$. This suggests that the linearization gradients in the quasilinearization algorithm may not be valid. If that is the case, then the linear expansion about the known iterate cannot be expected to result in an improvement. A modification developed by Ghosh (1973) to inhibit instabilities associated with too-large improvements has no effect upon these numerically divergent cases. Removal of this possible source of numerical instability further implies the difficulty is related to the linearity assumptions.

An explanation for the increasing slope is available from the differential equation for the adjoint variables, where

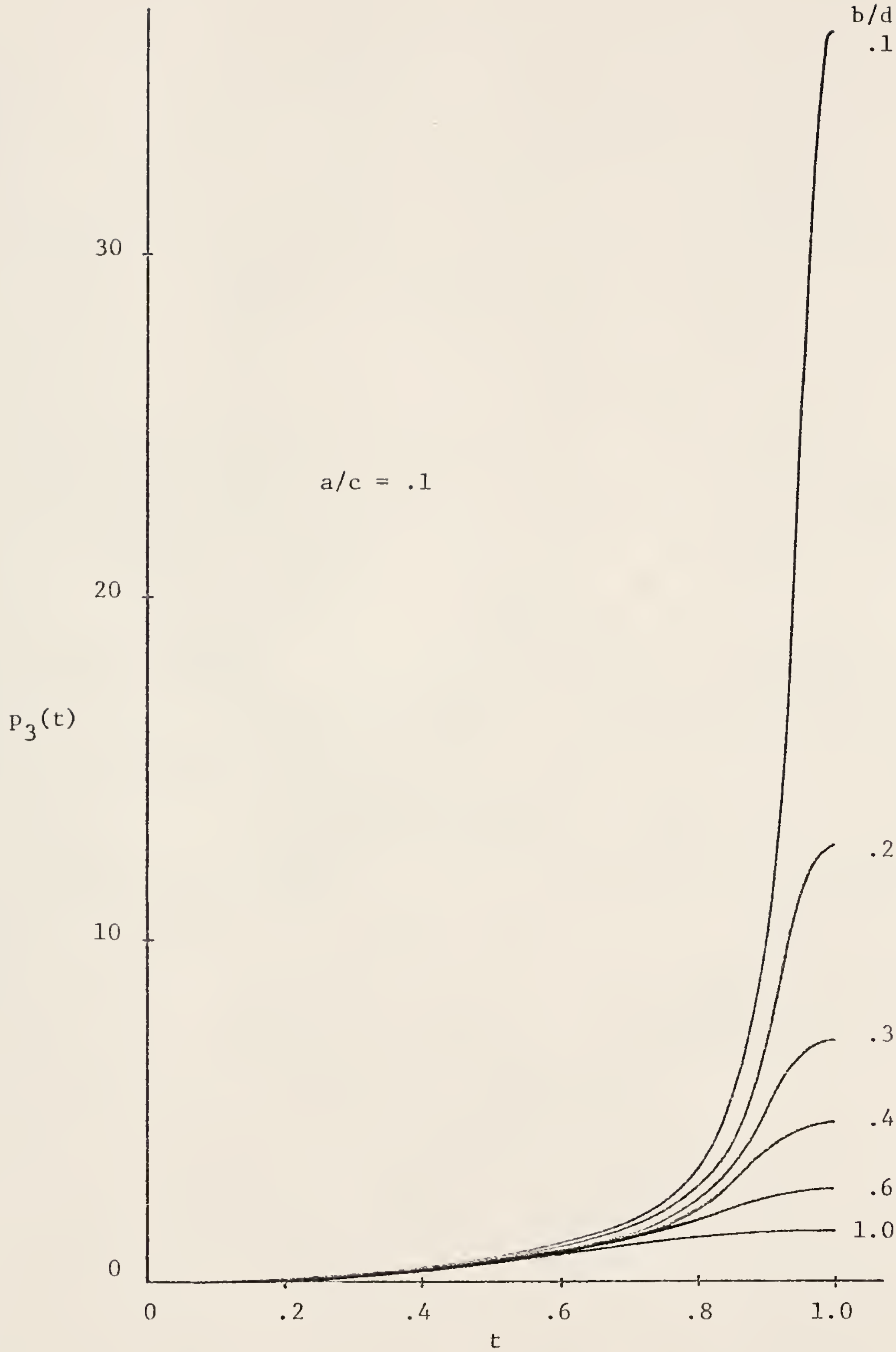

Figure 7.1 Solutions for Adjoint Variable $p_3(t)$ for Certain Cases of Interest

$$\dot{p}_3 = (1-t)/u_1{}^3u_2 \qquad p_3(0) = 0$$
$$\dot{p}_4 = -p_3 \qquad p_4(0) = 0 \tag{7.2.4}$$

Since both u_1 and u_2 are equal to their minimum allowable values near $t = 1$, $\dot{p}_3$ increases with decreasing parameter values in this region. However, the term $(1-t)$ forces $\dot{p}_3$ to zero at $t = 1$. For unknown reasons, the large curvature associated with this effect seemed to pose no difficulties for the numerical algorithm. As seen in equations (7.2.4), if $p_3(t)$ can be determined, then $p_4(t)$ is readily obtained. Curves for $p_4(t)$, corresponding to cases presented in Figure 7.1, are given in Figure 7.2. All six cases exhibit positive curvature and possess none of the inflection points seen in their counterparts of Figure 7.1. The increase in amplitude of these two variables for decreasing values of b/d is illustrated in Figure 7.3. In this figure it is seen that with respect to increasing amplitude as $b/d \to 0$, $p_3(t)$ is the critical variable.

The major disadvantage of the indirect solution method is that little is known about the connection between convergence and requirements upon the initial guess. For this specific example, only one of the seven variables causes convergence difficulties, and the resulting numerical instability occurs in a region of large, rapidly changing curvature. No convergence problems are encountered when the curvature of the initial guess more closely resembles the final solution. All of this suggests that the question of whether or not an initial guess allows the quasilinearization algorithm to converge, may depend upon the respective curvatures of the initial guess and the solution.

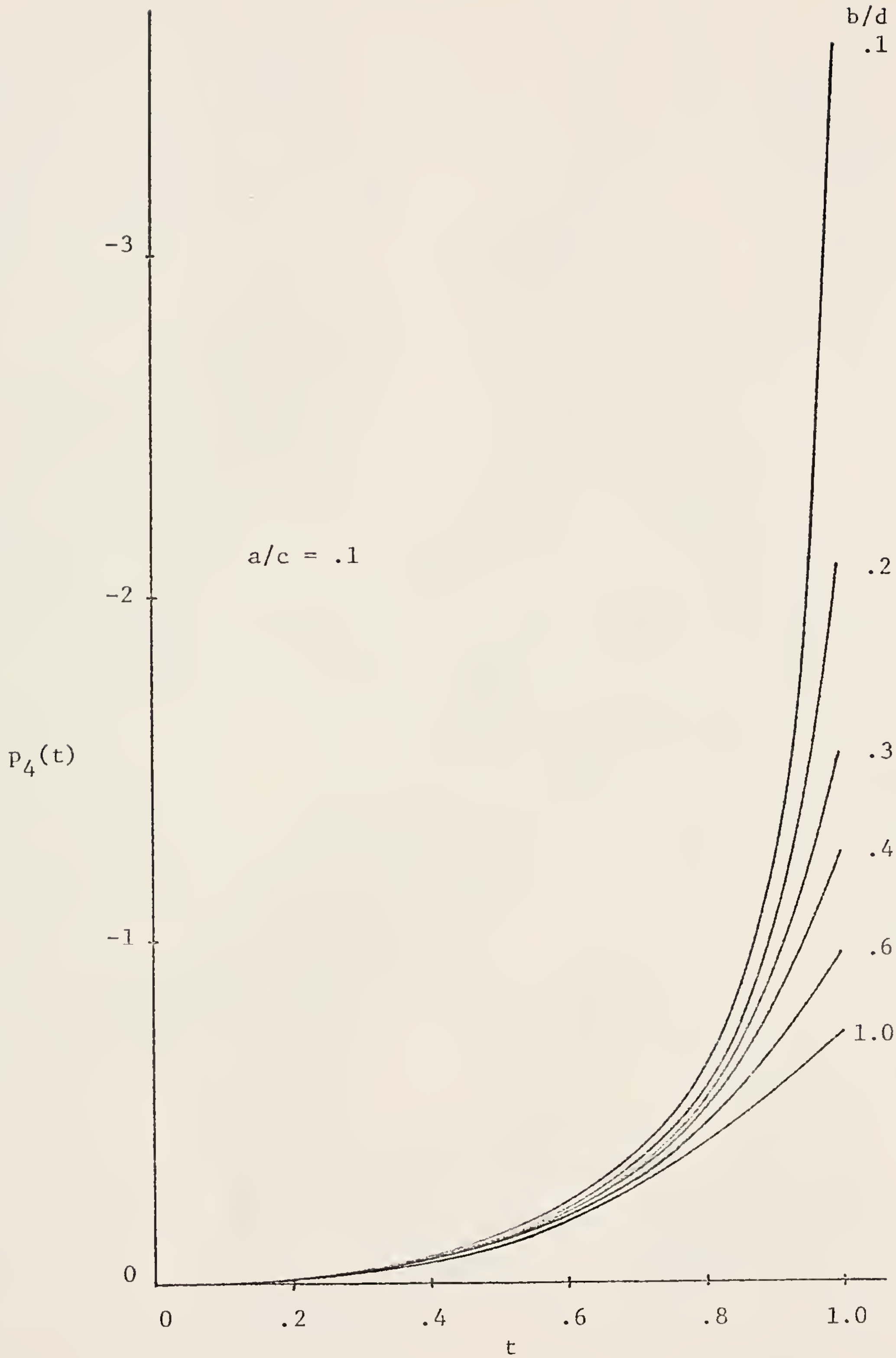

Figure 7.2 Solutions for Adjoint Variable $p_4(t)$ for Certain Cases of Interest

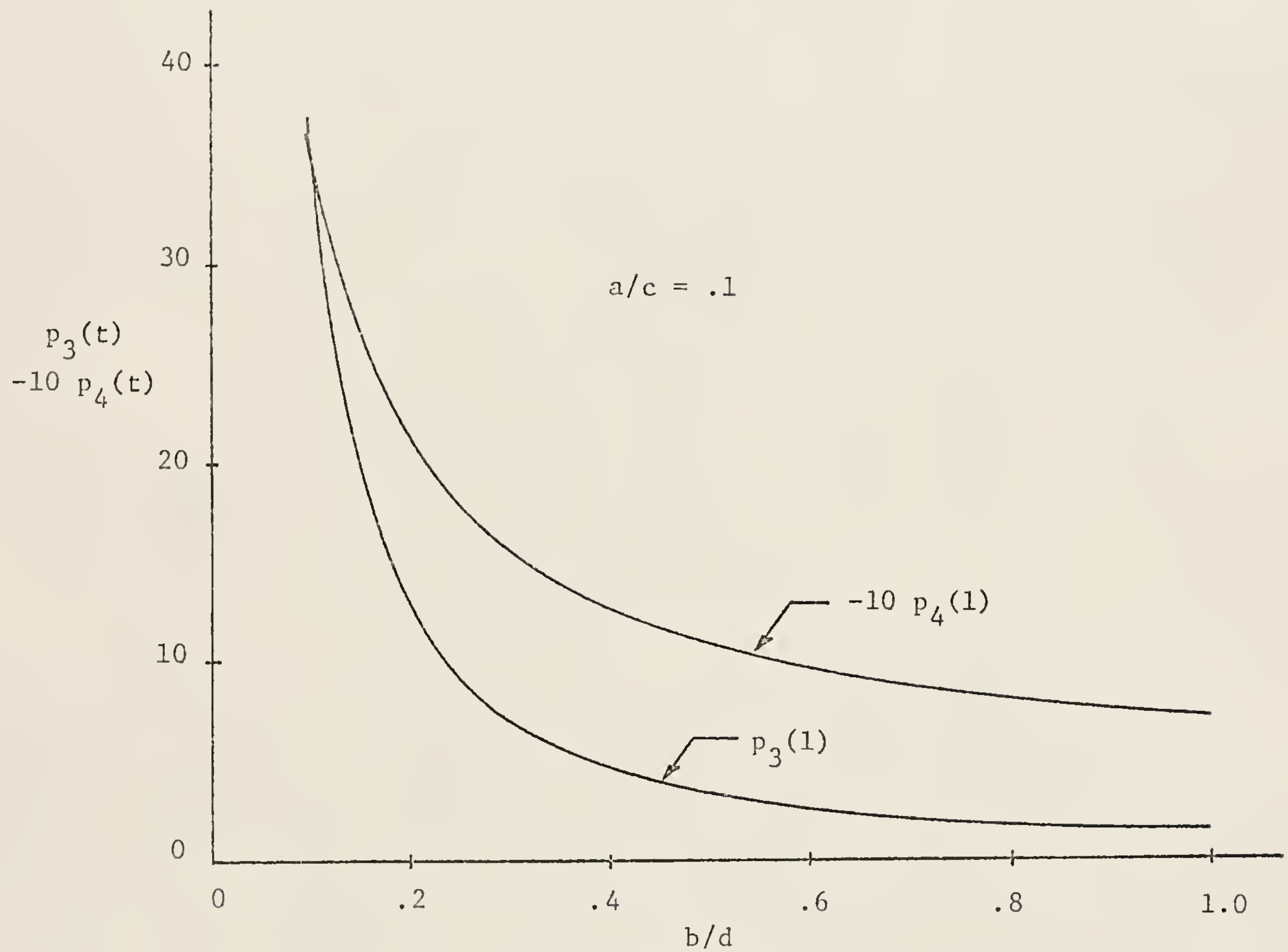

Figure 7.3 Terminal Point Amplitude of $p_3(t)$ and $p_4(t)$: Dependence upon Parameter b/d Value

7.3 Numerical Instabilities for Column Buckling Example

Another type of convergence failure was experienced with the column buckling problem. In these cases which did not converge, each of the variables was divergent, increasing with each iteration beginning with the initial guess. All of these numerically unstable cases were for parameter values which resulted in a solution that approached the uniform column limiting solution. Attempts to force convergence were unsuccessful.

The first attempt was to improve the initial guess used as a starting point by the algorithm. Although this technique succeeded with the beam problem, it failed with the column problem. The next attempt to obtain convergence was to employ the instability inhibiting modification developed by Ghosh. The only result was to prolong the numerical divergence. At this point it was suspected that perhaps the source of difficulty was related to normalization of the eigenfunction. However, when each iterate was normalized as indicated in section 5.4, the sole result was that the state variables maintained reasonable amplitudes while the amplitudes of the adjoint variables increased more than without normalization. Doubling the number of intervals in the finite difference approximation (from 100 to 200) was also ineffective.

A possible source of the numerical difficulty was the matrix inversion. For those cases failing to converge each of the $\underline{D}_i$ matrices was inverted accurately but the inversion of $\underline{C}$ was questionable, as indicated by the value of SUM. Since each iterate depends directly

upon the inverse of $\underline{C}$, a degradation in $\underline{C}^{-1}$ would cause a corresponding degradation in the "solution" at each iteration.

To illustrate what happens in these cases consider a specific example: $a_L = 0$, $a_U = 1.1$. A plot of values of SUM versus t_U, the intercept of the upper bound, is shown in Figure 7.4. These data points represent the $\underline{C}^{-1}$ accuracy at different iterations for various initial guesses--acceptable accuracy is indicated by the straight line. Basically, if an initial guess yielded a t_U of approximately one-half, each iteration gives an increased value of t_U with a correspondingly less accurate $\underline{C}^{-1}$. As t_U approaches one, the solution approaches the limiting uniform column.

The result of this is that $\underline{C}$ may become ill-conditioned if the parameters are chosen such that the solution is similar to a uniform column. There are no explicit data or derivations to indicate this is the situation, only the implicit suggestion of the data in Figure 7.4. However, as stated in the preceding section, there is little known about why certain initial guesses fail to converge in the indirect solution algorithms.

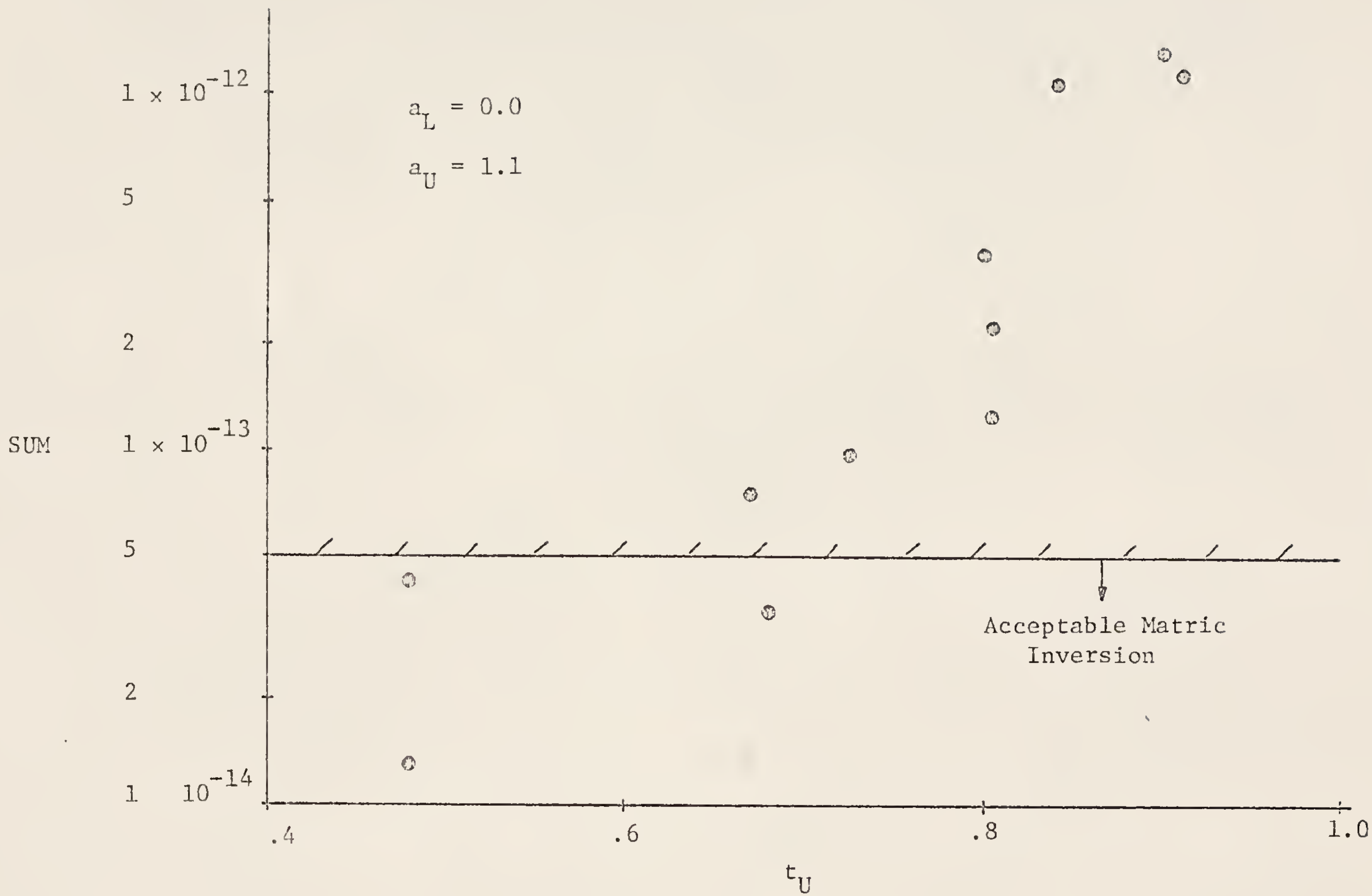

Figure 7.4 Degradation of C^{-1} Accuracy as Iterate Approaches a Solution Similar to the Uniform Column Limiting Solution

CHAPTER VIII

CONCLUSIONS AND RECOMMENDATIONS

8.0 Summary and Conclusions

This dissertation treats the optimal design of elastic structures subject to both hard inequality constraints and subsidiary conditions. It is shown that structural problems can be classified into two general types depending upon whether or not the cost functional is an eigenvalue. For the conservative systems considered, which are described by self-adjoint differential equations, it is shown that the minimum weight problem is identical to the maximum buckling load problem.

Pontryagin's Maximum Principle is analyzed as a nonlinear programming problem in Chapter III. Specifically, the theory of the gradient projection method is applied to the maximum principle, with the control subject to inequality constraints. From this an explicit formulation is obtained for the Lagrangian multipliers that adjoin the constraints to the variational Hamiltonian.

Additional characteristics of the maximum principle result from the gradient projection method analogy. In the latter, the projected gradient is that specific direction which minimizes the cost without violating the constraints in effect at any point in question. As applied to the maximum principle, the projected gradient is $H^*_{\underline{u}} = -(H + \underline{\mu}^T\underline{\phi})_{\underline{u}}$ which in $\underline{u}$-space denotes the direction of constrained

maximum descent whenever $\underline{\mu}$ is given by equation (3.3.4). While this attribute is not pursued further in the dissertation, it may prove useful in subsequent applications of the maximum principle. To further illustrate the nonlinear programming principles used in the theoretical development, the example of Chapter IV is approximated by finite elements, and solved by a gradient projection algorithm. Operation of the algorithm is illustrated and the results compared to those obtained with the maximum principle. When a sufficient number of finite elements are used to adequately describe the structure, the results of the two methods are identical.

A problem concerning the minimum deflection of a beam is optimized by using classical maximum principle techniques. It is shown that no finite solution is obtained for unconstrained control; this also holds for the column buckling problem treated in Chapter V. Furthermore, it is shown that for the conservative system corresponding to the buckling problem, the minimum weight and maximum buckling problems are identical. Another result of using the maximum principle is that for the problem described, in order to have a nontrivial solution the modulus and density must be the maximum and minimum allowable constant values, respectively. This is a mathematical verification of an intuitive result.

Numerical instabilities experienced by the quasilinearization algorithm are detailed in Chapter VII. While no concrete results are obtained, from the data presented it appears that the question of convergence is related to the respective curvatures of the initial guess and the solution.

8.1 Recommendations

Further work suggested in the course of dissertation research falls into three categories: structural considerations, mathematical theory, and numerical solution techniques. Under the first category the most obvious extension is to not invoke the linear bending assumption. Since the quasilinearization algorithm handles nonlinear TPBVP, theoretically there is no need for the assumption. In contrast to the first of equations (5.2.4) the complete, nonlinear bending problem is described by

$$(s(t)\ddot{\eta}) = M(t)[1 + \dot{\eta}^2]^{3/2} \qquad , \quad 0 \le t \le 1$$

$$M(t) = \lambda[\eta(1) - \eta(t)] + k \int_t^1 m(\xi)[\eta(\xi)-\eta(t)]d\xi$$

$$\eta(0) = \dot{\eta}(0) = 0$$

Another interesting possibility is to include the effect of shear.

These structural aspects pertain to specific problems. A more general result, and hence more interesting, is related to expanding the class of problems that can be treated by energy techniques. It is suggested that the mutual potential energy method and the application of adjoint systems to nonconservative problems be investigated to see if a "general energy method" exists. Such a development would expand the class of problems now considered to be amenable to the energy method and its well-developed theory.

In terms of mathematical theory, the double optimization problem associated with extremization of eigenvalues for self-adjoint systems is an inviting prospect. No significant contributions have been

published in recent years. Any new discovery of a general nature would be a significant development. Another useful area of mathematical research would be the further study of the nonlinear programming aspect of the maximum principle. Beyond this, it might also prove fruitful to combine the latter with a comparison to dynamic programming.

Finally, the discussion of numerical instabilities of Chapter VII indicates the possibility that the respective curvatures of the initial guess and the solution determine convergence. Any development that could indicate convergence, or the lack thereof, for an initial guess required by an indirect solution technique, would be a significant development. One possibility is a study of the sensitivity of the solution increment function, $\underline{\varepsilon}(t)$, to error sources such as initial guess, step size, etc., for a case with a known solution. Little is known about this subject.

APPENDIXES

APPENDIX A

HISTORICAL DEVELOPMENTS

The dissertation subject matter originated from two papers: a short survey of structural optimization problems by Prager and Taylor (1968) and a technical note by Boykin and Sierakowski (1972). A survey of the literature began with the former and revealed that what were presumed to be recent developments were in fact two hundred years old. Besides engendering a real sense of humility, my discovery aroused a fear that some fundamental part of my work might be a duplication of a much earlier study. I also found the historical development to be quite intriguing; for that reason this appendix has been included to give the reader a brief history of the separate development of mechanics and the calculus of variations. Neither is intended to be comprehensive; however, the reader may enjoy being able to associate some recognizable equation or method with a specific person and period of time.

The appendix consists of three sections, where the first is an anecdotal discussion of the major early contributors to the calculus of variations and a table listing major developments in chronological order. This is followed by a similar account for mechanics, also having a table of chronologically ordered developments. Concluding the appendix is a short biographical table that lists the life span and nationality of most of the people mentioned.

Newton posed a problem in Book II of the Principia which requires the techniques of the calculus of variations for solution. His goal was to find the solid of revolution which has the minimum resistance in axial flow. No apparent significance was attached to the problem of either Newton or his contemporaries.

The calculus of variations actually begins with two Swiss mathematicians, James Bernoulli and his younger brother John. Until 1690 the latter was a student of James, but soon became a rival. In June, 1696, John Bernoulli posed the brachistochrone problem in Acta Eruditorum:

> A New Problem, to the Solution of Which
> Mathematicians Are Invited
>
> Given two points A and B in a vertical plane, to find for the movable (particle) M, the path AMB descending along which by its own gravity, and beginning to be urged from the point A, it may in the shortest time reach the other point B.

According to various sources, solutions were offered by Newton, Leibnetz, de l'Hopital, and the Bernoullis. At Leibnetz' request the solution was withheld to encourage others to consider the problem. In January, 1697, the problem was reannounced. In the following May issue of Acta Eruditorum, the solutions of the Bernoullis and the Marquis de l'Hopital were published. John's is alleged to be the most readable but since it is solved as an analog to an optics problem, the method is restricted to a small class of problems. The solution of James is quite geometrical, treating the curve as a polygon with sides of infinitesimal length. It was also assumed that whatever optimal

property the entire curve possessed, was also possessed by each part. The solution of de l'Hopital was given, but without proof.

At the conclusion of his paper, James posed two more difficult problems:

> First: find the curve of quickest descent from a fixed point A to a given vertical line. This version of the brachistochrone problem involves conditions for minimizing a definite integral where variations at the limits are permitted. Such conditions lead to natural boundary conditions, or transversality conditions, and were not obtained until some years later by Lagrange.
>
> Second: among all curves with given length and base, find the the curve such that a second curve, whose coordinates are some function of the first, contains a maximum or minimum area. This subsidiary condition is called an isoperimetrical constraint.

Besides inviting all mathematicians to attempt solutions, James specifically offered his brother John a prize of fifty ducats for the correct solution--according to Bliss (1925, p. 12). John claimed the award but James refused, showing that the assumption of a uniform optimality property for each infinitesimal element was not valid. James published the correct solutions in the May, 1701, issue of Acta Eruditorum; to include the effect of the constraint, he used three adjacent polygon elements of infinitesimal length instead of two that sufficed for the brachistochrone, and no fewer than four solution techniques:

(1) Equilibrium of forces on an element

(2) Equilibrium of moments on an element

(3) Principle of virtual work

(4) Principle of minimum potential energy

Whether or not the prize existed in immaterial; it is true the problem embroiled the brothers in a bitter quarrel, and together with the brachistochrone problem marks the beginning of the calculus of variations.

All of this work by the Bernoullis was based upon complicated geometrical results. A student of theirs, Leonhard Euler, expanded and generalized their initial developments. His first major contribution to the calculus of variations was the differential equation bearing his name. Euler suggested that for continuous functions a definite integral vanishes identically if the integrand vanishes. When the integrand is the product of some function $M(x,\dot{x},t)$ and an arbitrary variation $\delta x(t)$ the function must vanish if the variation is to be arbitrary. Euler's equation is $M(x,\dot{x},t) = 0$; up to the first part of this century this vanishing integrand was referred to as "Euler's Fundamental Lemma"--"Euler's equation" is the more recent appellation. He also was the first to indicate the two general classes of optimal solutions obtained from his technique: absolute and relative. After reading a memoir by Lagrange which introduced the symbol "δ," Euler adopted its use for his own studies and named it in an article entitled "Elementia Calculi Variationum." Other investigations related to this include combining the calculus techniques of Newton and Leibnetz via differential elements, and what is apparently the earliest study of integrability. Euler's list of 886 books and articles also includes the introduction of "Σ" to denote summation, and an intuitive description of continuous functions--all of which are necessary elements of the calculus of variations.

As mentioned in the previous paragraph, Lagrange introduced the concept and symbol "δ"; using it enabled him to derive Euler's equation

without resorting to the infinitely-sided polygon and the accompanying complicated geometry. It also facilitated the evaluation of problems having definite integrals with unspecified conditions at the limits--these were called "the definite equations," and later "terms at the limits," but are currently referred to as "natural boundary conditions." This represented a generalization on the type of problem originally posed by James Bernoulli. Lagrange introduced yet another generalization by using "indeterminate multipliers" to enforce implicit constraints, e.g., differential equations. The final major contribution by Lagrange to be cited here was the extension to problems having double integrals with fixed conditions at the limits.

Following a study of the major works of Newton, Laplace, and Lagrange, William Rowan Hamilton applied the analytical method of Lagrange to his own study of optics. In the "Theory of Systems of Rays" (1828) and its supplements, Hamilton deduced all optical properties from a single "Characteristic Function" using the "Law of Varying Action," a generalization of the "Law of Least Action" due to P. L. M. de Maupertis. When applied to dynamical systems in "On General Methods of Dynamics" (1834) this method is recognized as Hamilton's Principle. The canonical equations of motion derived from this method are called Hamilton's equations, and were used by Jacobi to later derive the Hamilton-Jacobi equation.

Euler derived the first necessary condition for an optimal solution in the calculus of variations: in the manner developed by Lagrange, the first variation must vanish. But this may be satisfied by either

a minimum, maximum, or an "inflection point." Legendre made the first attempt to distinguish what the case may be by investigating the second variation in 1786. Using a transformation of variables, he arrived at what was believed to be a sufficient condition (for either a maximum or a minimum) for fifty years. However, in 1837 Jacobi discovered that Legendre's condition failed in some cases. The "conjugate points" associated with these cases were found from derivatives of the solution to Euler's equation with respect to the constants of integration. As a result, Legendre's condition was determined to be only necessary, along with the newer Jacobi's condition, and not sufficient as originally thought.

Derivation of a true sufficient condition was done by Weierstrass in 1879. It occurred as a result of his re-examination of earlier works, formulating the problems very precisely. Both his condition and his "corner condition" for a discontinuous solution are stated in terms of the "Weierstrass function."

Further elaborations, involving geodesics and the existence of integrals, are attributed to Kneser and Hilbert about 1900. Twenty years later Bliss applied the "adjoint system" to the solution of a variational problem. The "maximum principle" was suggested in 1937 by Valentine but is generally attributed to Pontryagin in 1956.

All of these basic developments are presented in Table A.1, together with less pertinent details. According to Bliss (1925, p. 181) the classical memoirs of the Bernoullis, Euler, Lagrange, Legendre, and Jacobi are contained in Ostwald's "Klassiker der exacten Wissenschaften,"

TABLE A.1

MAJOR DEVELOPMENTS IN THE CALCULUS OF VARIATIONS

1686	Newton	...	first to pose a problem of a type requiring the calculus of variations; not treated as significant at the time.
1696	John Bernoulli	...	posed the brachistochrone problem. Solutions were obtained by James Bernoulli, Leibnetz, and Newton, as well as by John Bernoulli and de l'Hopital. Accompanying his solution were two additional problems proposed by James Bernoulli, one involving natural boundary conditions, another with an isoperimetrical constraint.
1701	James Bernoulli	...	solved the isoperimetrical constraint problem posed in his solution to the brachistochrone.
1722	Euler	...	extended the work of the Bernoulli brothers to several classes of problems with various constraints.
1740	Euler	...	introduced the technique that setting coefficient of δY in integrand to zero satisfies necessary condition. The result is called Euler's Equation.
1744	Euler	...	divided optimization problems into two general classes: absolute and relative. This is the first systematic exposition; all preceding work represented ad hoc solutions to specific problems.
1766	Euler	...	published one tract giving the name "calculus of variations," and another containing the first study of conditions of integrability.
---*	Lagrange	...	introduces the symbol "δ" to distinguish between variation and derivative.
---*	Lagrange	...	deduced Euler's Equation without using adjacent elements of an infinitely-sided polygon.

*No specific dates are available; two memoirs on the general subject from which these might originate were published in 1762 and 1770.

TABLE A.1 (Continued)

---*	Lagrange	...	determined the method to treat unspecified end conditions; the results are now called natural boundary conditions.
---*	Lagrange	...	introduced enforcement of implicit constraints by the use of indeterminate multipliers.
---*	Lagrange	...	extended the calculus of variations to include double integrals with fixed boundary conditions.
1786	Legendre	...	investigated the second variation of the cost function to find a criterion with which maxima and minima could be distinguished. First results were not really a sufficient condition, because of non-unique solutions demonstrated by Lagrange
1810	Brunaci	...	extended Legendre's work to double integrals, retaining the same general flaw.
1834	Hamilton	...	developed the canonical equations and Hamilton's Principle based upon a generalization of the Law of Least Action.
1837	Jacobi	...	studied the transformation of Legendre, discovering how to determine when the method failed. The resulting conjugate points are determined from derivatives of solutions to Euler's equation with respect to the constants of integration.
1842	Sarrus	...	won the French Academy of Science prize for mathematics by obtaining the natural boundary conditions for a general multiple integral having variable limits. His technique used a new substitution sign to designate a particular value for general variable.
1844	Cauchy	...	simplified the work of Sarrus to the form used currently.

*No specific dates are available; two memoirs on the general subject from which these might originate were published in 1762 and 1770.

TABLE A.1 (Concluded)

1852	Brioschi	...	was the first to apply determinants to the investigation of second order terms.
1858	Clebsch	...	generalized Jacobi's Theorem to several dependent variables, with or without connecting equations, and for multiple integrals.
1871	Todhunter	...	suggested that variations might be of restricted sign, allowing the possibility of discontinuous solutions.
1877	Erdmann	...	derived the conditions that must be satisfied by a discontinuous solution, called the Weierstrass-Erdmann conditions.
1879	Weierstrass	...	helped to make the theory of the calculus of variations more precise, in general. Specifically, he found a new necessary condition in terms of his function E(X,Y,P,Y') and very clearly determined those conditions which are sufficient.
1919	Bliss	...	was perhaps the first to use the adjoint system of variational equations.
1937	Valentine	...	derived the earliest equivalent to a strong minimum principle, using the Weierstrass condition and slack variables.
1956	Pontryagin	...	together with colleagues, derived the Pontryagin maximum principle.

Nos. 46 and 47. Historical developments are treated by Carll (1881), Todhunter and Pearson (1893), Bolza (1904), and Bliss (1925, 1946). Biographical data are found in Truesdell (1968), Bergamini (1963), Ireland (1962), and Asimov (1960).

Analytical mechanics became a viable method with the introduction of calculus by Leibnetz and Newton. For example, Galileo Galilei observed in 1585 that the free fall of various objects of different size and weight is given by the formula $Y = 16t^2$. Newton is reported to have differentiated twice to discover that all objects fall with an acceleration of thirty-two feet per second squared. Yet there were differences between the calculus of Newton and Leibnetz; the former designated a derivative by a dot above the variable, and the latter used the form "dY/dt" to indicate the derivative of "Y" with respect to "t." Newton also deduced that maxima/minima occur at points where the rate of change is zero, whereas Leibnetz insisted such points occurred where the tangent to the curve has zero slope.

To show that the work of these two was equivalent was one of the many accomplishments of Euler. In using this discovery he demonstrated that much of what was then considered to be pure physics was transformed quite simply into problems of mathematics. Such work led him to sufficiently important developments that specific equations bear his name in the separate fields of the calculus of variations, differential equations, solid mechanics, and rigid body dynamics. According to Truesdell (1968), Euler is the major source of rational analysis. Certainly he is one of the most prolific, having published 886 books and articles, and fathering thirteen children.

A more detailed list of the basic developments of mechanics is given in Table A.2; a few biographical data are presented in Table A.3.

TABLE A.2

MAJOR DEVELOPMENTS IN ANALYTICAL MECHANICS

1694	James Bernoulli	...	published the very first paper on the mathematical theory of elasticity.
1704	James Bernoulli	...	concluded a thirteen year study which included an investigation of the catenary (results were published in 1744). Problem was also solved by John Bernoulli using a theorem concerning the center of gravity of an arc.
---*	James Bernoulli	...	declared elasticity is generally non-linear, with linear elasticity a special case. He also derived the differential equation for the elastica.
1704	James Bernoulli	...	first characterized a material with stress as a function of strain instead of the then accepted load versus displacement for a particular specimen. He avoided the E of linear elasticity, suggesting E_t by implication.
---*	Leibnetz	...	performed the first analysis of a loaded beam, assuming that some fibres were in tension. That the tension varied linearly gave the result that the moment was proportional to the second area moment of the cross section.
1727	Euler	...	derived the Bernoulli-Euler equation for bending of beams while a student of John Bernoulli. The work was not published until 1862.

*No specific dates are available from ordinary sources; items are located with respect to related entries of the table.

TABLE A.2 (Continued)

1729	Musschenbroek	...	performed the first comprehensive experiments on the strength of materials. He observed that beams and columns bend before breaking and that breaking strength varies inversely with the square of the length and also depends upon depth and breadth.
1736	Euler	...	introduced the study of mechanics, specifically the concepts point mass, acceleration, and vectors (which he called "geometrical quantity").
1739	Daniel Bernoulli	...	developed the principle of minimum potential energy, although he is best known for his work in fluid mechanics.
1740	Euler	...	learned how to solve analytically, linear ordinary differential equations with constant coefficients. Prior to that time only series solutions were available.
---*	Euler	...	after making some calculations on an actual beam deflecting under its load, discovered that most real applications result in small strain. All previous work had presumed that deformations were finite.
1743	Euler	...	determined that the equation of the elastic curve could be obtained from the minimum potential energy.
1743	Euler	...	derived the column buckling formula; the results were compared to Musschenbroek's experiments in 1765.
1743	John Bernoulli	...	obtained the first differential equation for the motion of a system by studying the dynamics of a weighted cord using lumped masses.

*No specific dates are available from ordinary sources; items are located with respect to related entries of the table.

TABLE A.2 (Continued)

1743	D'Alembert	...	introduced the first partial differential equations. He also was the first to separate constraint forces from external forces, but never stated "D'Alembert's Principle," which was given later by Euler and Lagrange.
1744	Euler	...	published the earliest example of what came to be called the "Newtonian Method." Also defined were linear momentum and kinetic energy derived by integration.
1747	Euler	...	was the first to derive "Newton's Equations" for a discrete system. By applying them to each part of the system, all the required equations were obtained. The example used was the three-body problem of celestial mechanics.
1750	Euler	...	extended the preceding study to "mechanical systems of all kinds," i.e., to continuous systems. Examples used included rigid body motion and transverse vibrations of a rod.
1754	Riccati	...	suggested that the work done to deform an elastic body was recoverable in part.
1760	Euler	...	developed an entire theory of rigid body motion. He defined center of mass and distinguished it from center of gravity. Additionally, he obtained the inertia tensor and described its elements with respect to the mass.
---*	Lagrange	...	determined the infinite sequence of theoretical buckling loads.
1773	Lagrange	...	attempted to find the column shape which would have the largest buckling load for given height and volume. His analysis was incorrect; this results, a circular cylinder.

---

*No specific dates are available from ordinary sources; items are located with respect to related entries of the table.

TABLE A.2 (Concluded)

1778	Euler	...	determined the height at which a uniformly heavy column buckles.
1788	Lagrange	...	published one of the most important works in mechanics: "Méchanique Analitique." In this book he systematized mechanics, using the calculus of variations to derive general equations which apply to all problems of mechanics. Generally speaking, he obviated the use of complicated geometry, replacing it with pure algebra. More specifically, he introduced the misnomer "D'Alembert's Principle," generalized coordinates, and Lagrange's equations.
1877	Gauss	...	proved the fundamental theorem of algebra in his doctoral dissertation, requiring the introduction of complex numbers. He later recognized their vector character and applied them to the solution of complicated mechanics problems.

TABLE A.3

MAJOR CONTRIBUTORS IN THE DEVELOPMENT OF THE CALCULUS OF VARIATIONS AND ANALYTICAL MECHANICS

NEWTON, Sir Isaac	(1642-1727)	English mathematician
LEIBNETZ, Baron von Gottfried Wilhelm	(1646-1716)	German mathematician
BERNOULLI, John (Jean, Johan)	(1667-1748)	Swiss mathematician
BERNOULLI, James (Jacques, Jakob)	(1654-1705)	Swiss mathematician
MUSSCHENBROEK, Pieter van	(1692-1761)	Dutch physicist and mathematician
BERNOULLI, Daniel	(1700-1782)	Swiss mathematician
EULER, Leonhard	(1707-1783)	Swiss mathematician
ALEMBERT, Jean Le Rond D'	(1717-1783)	French astronomer and mathematician
LAGRANGE, Comte Joseph Louis	(1736-1813)	French mathematician
LEGENDRE, Adrien Marie	(1752-1833)	French mathematician
GAUSS, Karl Friedrich	(1777-1855)	German astronomer and mathematician
HAMILTON, Sir William Rowan	(1805-1865)	Irish mathematician
JACOBI, Karl Gustav Jacob	(1804-1851)	German mathematician
BRIOSCHI, Francesco	(1824-1897)	Italian mathematician
CLEBSCH, Alfred Rudolf Friedrich	(1833-1872)	German mathematician
TODHUNTER, Isaac	(1820-1884)	English mathematician and historian
WEIERSTRASS, Karl Theodor	(1815-1897)	German mathematician
BLISS, Gilbert Ames	(1876-1951)	American mathematician
PONTRYAGIN, Lev S.	(1908-)	Russian mathematician

APPENDIX B

A SIMPLE PROOF OF THE KUHN-TUCKER THEOREM

Consider the minimization of a general unspecified functional with respect to an n-dimensional vector $\underline{x}$, subject to $m < n$ inequality constraints,

functional: $\min_{\underline{x}} [F(\underline{x})]$

inequality constraints: $g_j(\underline{x}) \leq 0 \qquad j = 1,\ldots,m$

where

$$\underline{x} = [x_1 \; x_2 \; \ldots \; x_n]^T$$

Inequality constraints are converted to equality constraints by introducing "slack" variables $s_j(\underline{x})$ defined such that

$$g_j(\underline{x}) + s_j^2(\underline{x}) = 0 \qquad j = 1,\ldots,m \tag{B.1}$$

or

$$s_j(\underline{x}) = [-g_j(\underline{x})]^{\frac{1}{2}} \geq 0 \tag{B.2}$$

Since for all j the left-hand side of equation (B.1) sums to zero, for arbitrary values of λ_j

$$\sum_{j=1}^{m} \lambda_j [g_j(\underline{x}) + s_j^2(\underline{x})] = 0 \tag{B.3}$$

it follows that:

$$\Phi(\underline{x},\underline{s},\underline{\lambda}) = F(\underline{x}) + \sum_{j=1}^{m} \lambda_j [g_j(\underline{x}) + s_j^2(\underline{x})] \equiv F(\underline{x})$$

Necessary conditions for the adjoined cost function Φ to be a minimum are derived from the requirement that the first variation vanish.

$$s\Phi = \delta\underline{x}^T \frac{\partial\Phi}{\partial\underline{x}} + \delta\underline{s}^T \frac{\partial\Phi}{\partial\underline{s}} + \delta\underline{\lambda}^T \frac{\partial\Phi}{\partial\underline{\lambda}} = 0$$

If all the variations are to be independent, then

$$\nabla_{\underline{x}} F(\underline{x}) + \lambda_j \nabla_{\underline{x}} g_j(\underline{x}) = 0 \qquad j \text{ summed}$$

$$2\lambda_j \, s_j(\underline{x}) = 0 \qquad j = 1,\ldots,m$$

$$g_j(\underline{x}) + s_j^2(\underline{x}) = 0 \qquad j = 1,\ldots,m$$

where repeated indices in a product imply summation. The independence of $\underline{s}$ and $\underline{\lambda}$ can be argued heuristically. Slack variables are an artificial contrivance used to force a desired condition without altering the original problem. Any change in $s_j(\underline{x})$ which satisfies the given constraint must be accompanied by a change in $g_j(\underline{x})$ such that the net effect on Φ is zero; thus $\Phi(\underline{x},\underline{s},\underline{\lambda})$ is stationary with respect to $\underline{s}$. Furthermore, since the sum in (B.3) is equal to zero for any arbitrary values of λ_j, any arbitrary variation of Φ with respect to $\underline{\lambda}$ must also be zero.

Given the definition of $s_j(\underline{x})$ in (B.2), the three necessary conditions may be expressed as

$$\nabla_{\underline{x}}[F(\underline{x}) + \lambda_j g_j(\underline{x})] = 0 \tag{B.4}$$

$$\lambda_j g_j(\underline{x}) = 0 \tag{B.5}$$

$$g_j(\underline{x}) \leq 0$$

The first condition will be used with the concept of a feasible, usable direction to derive the final condition remaining before the Kuhn-Tucker condition may be stated. Any direction in $\underline{x}$-space is feasible if an increment $\Delta\underline{x}$ in that direction reduces the cost function $F(\underline{x})$. Direction $\hat{d}$ is said to also be usable if an increment $\Delta\underline{x}$ in that direction causes no constraint violation.

Consider first the concept of a feasible direction; for any direction $\hat{d}$ the directional derivative is defined to be

$$\frac{dF}{ds} = \hat{d} \cdot \nabla_{\underline{x}} F(\underline{x})$$

where

$$||\hat{d}\,|| = 1$$

In addition to this restriction on the norm of direction $\hat{d}$, let the increment $d\underline{x}$ be prescribed by

$$d\underline{x} = (ds)\hat{d} \tag{B.6}$$

and the vector scalar product be written as the equivalent inner product such that

$$\hat{d} \cdot \nabla_{\underline{x}} F \equiv \hat{d}^T \nabla_{\underline{x}} F \equiv (\nabla_{\underline{x}} F,\ \hat{d})$$

In terms of an inner product notation

$$\frac{dF}{ds} = (\nabla_{\underline{x}} F,\ \hat{d})$$

For an increment in direction $\hat{d}$ of length ds, the change in the cost function is

$$dF = (\nabla_{\underline{x}} F,\ \hat{d})\ ds$$

By the linearity property of the inner product

$$dF = -\ (-\nabla_{\underline{x}} F,\ \hat{d})\ ds$$

Hence the cost $F(\underline{x})$ is decreased by an increment in any direction $\hat{d}$ which satisfies

$$(-\nabla_{\underline{x}} F,\ \hat{d}) > 0 \tag{B.7}$$

Conversely, a direction $\hat{d}$ is feasible if $(-\nabla_{\underline{x}} F,\ \hat{d}) > 0$, i.e., the angle between $\hat{d}$ and the direction of steepest descent is acute.

For any direction to be usable, an increment $\Delta\underline{x}$ in that direction must cause no constraint violation. On the constraint boundaries $g_j(\underline{x}) = 0$ for $j \in I_A$ as defined in Chapter III. With any direction again denoted by $\hat{d}$, the change in $g_j(\underline{x})$ with respect to an increment $d\underline{x}$ is defined through the directional derivative:

$$\frac{dg_j}{ds} = (\nabla_{\underline{x}}\ g_j(\underline{x}), \hat{d})$$

where

$$||\hat{d}\ || = 1$$

such that

$$dg_j = (\nabla_{\underline{x}}\ g_j,\ \hat{d})\ ds$$

Since ds denotes a vector norm by way of the definition of $d\underline{x}$ given in equation (B.6), it is a positive quantity. Hence the sense of change dg_j is prescribed by the inner product $(\nabla_{\underline{x}}\ g_j,\ \hat{d})$ for which there are three possibilities.

(i) $(\nabla_{\underline{x}} g_j, \hat{d}) < 0 \rightarrow dg_j < 0$

It was assumed that active constraints are being discussed, for which $g_j(\underline{x}) = 0$. The constraints require all $g_j(\underline{x})$ to be non-positive. An increment $d\underline{x}$ in the direction $\hat{d}$ used in the inner product satisfies the constraint with the inequality sign. Because the direction does not lead to a constraint violation it is by definition usable. Actually, for such a direction $\hat{d}$, even though point $\underline{x}$ lies on the boundary $g_j(\underline{x}) = 0$, the point $(\underline{x} + d\underline{x})$ does not.

(ii) $(\nabla_{\underline{x}} g_j, \hat{d}) = 0 \rightarrow dg_j = 0$

Under the assumption that $g_j(\underline{x})$ is active where $g_j(x) = 0$, when the increment $d\underline{x}$ in direction $\hat{d}$ causes no change in g_j, $\hat{d}$ is tangent to the constraint surface. With $\underline{x} + d\underline{x}$ used in the inner product, the point $(\underline{x} + d\underline{x})$ may still lie on the constraint boundary. With such an increment the constraint may perhaps not be violated, and if not, the direction is therefore usable. Note that for linear constraints, the word "may" is deleted from the previous sentence. It is included because nonlinear constraints require an argument calling for infinitesimal increments; a finite increment leads to a point that is not on the constraint boundary. A finite increment from a point on a nonlinear constraint boundary results in that constraint either becoming inactive on being violated, depending upon whether the constraint surface is concave or convex, respectively.

(iii) $(\nabla_{\underline{x}} g_j, \hat{d}) > 0 \rightarrow dg_j > 0$

That the constraint $g_j(\underline{x})$ is active requires $g_j(\underline{x}) = 0$. If an increment $d\underline{x}$ in the direction $\hat{d}$ used in the inner product results in a positive dg_j, the constraint has been violated. Therefore, $\hat{d}$ is not a usable direction.

The result of these possibilities is that for any point $\underline{x}$ on a constraint boundary, for a direction $\hat{d}$ to be usable it must satisfy

$$(\nabla_{\underline{x}} g_j, \hat{d}) \leq 0 \qquad\qquad j \in I_A \qquad\qquad (B.8)$$

If the set of all directions $\hat{d}$ which are feasible contains no usable directions at the point $\underline{x}$ in question, then $\underline{x} = \underline{x}_{OPT}$ and $\underline{x}$ is the <u>constrained</u> <u>minimum</u> <u>point</u>.

Conditions for a usable, feasible direction given by (B.8) and (B.7) are used with the gradient condition (B.4) to prove that the Lagrangian multipliers of (B.5) must be non-negative at the point where $\underline{x}$ is a constrained optimum. Condition (B.4) states that at the constrained minimum point, the negative gradient of the cost function lies in a subspace defined by the gradients of the active constraints, i.e.,

$$- \nabla_{\underline{x}} F(x) = \lambda_j \nabla_{\underline{x}} g_j(\underline{x})$$

In the following, implied summation is replaced by the symbol Σ, and the dependence of F and g_j upon $\underline{x}$ is implied. Resolving $-\nabla_{\underline{x}} F$ into components along the $\nabla_{\underline{x}} g_j$ directions via the inner product gives

$$-\nabla_{\underline{x}} F = \sum_{j=1}^{m} \frac{(-\nabla_{\underline{x}} F,\ \nabla_{\underline{x}} g_j)}{\|\nabla_{\underline{x}} g_j\|} \nabla_{\underline{x}} g_j$$

Thus

$$\lambda_j = \frac{(-\nabla_{\underline{x}} F,\ \nabla_{\underline{x}} g_j)}{\|\nabla_{\underline{x}} g_j\|}, \qquad j \in I_A$$

This condition relating the Lagrangian multipliers to an inner product is used with the conditions necessary for the existence of a constrained minimum to show that $\lambda_j > 0$. At a constrained optimal point, for every feasible direction $\hat{d}$, the cost decreases and

$$(-\nabla_{\underline{x}} F,\ \hat{d}) > 0 \leftarrow \quad \hat{d} \text{ is feasible}$$

the constraints are violated (otherwise not optimal),

$$(\nabla_{\underline{x}} g_j,\ \hat{d}) > 0 \leftarrow \quad d \text{ is not usable}$$

Assume that some j, λ_j is negative and a feasible but not usable direction $\hat{d}$ exists. If $\hat{d}$ is taken to be coincident with $-\nabla_{\underline{x}} F$, this direction is feasible and usable, contrary to the original assumption. That $\lambda_j > 0$ is proved by contradiction. Mathematically stated,

Given: $(-\nabla_{\underline{x}} F,\ \hat{d}) > 0$ for all $\hat{d}$

$(\nabla_{\underline{x}} g_j,\ \hat{d}) > 0$ for all $j \in I_A$

Assume: $\lambda_j < 0$ for some $j \in I_A$

(i) $\lambda_j < 0 \rightarrow (-\nabla_{\underline{x}} F,\ \nabla_{\underline{x}} g_j) < 0$

(ii) Consider $\hat{d} = -\nabla_{\underline{x}} F$

then

$$(-\nabla_{\underline{x}} F, \hat{d}) = (-\nabla_{\underline{x}} F, -\nabla_{\underline{x}} F) = \| -\nabla_{\underline{x}} F \|^2$$

$$> 0 \rightarrow \hat{d} \text{ is feasible}$$

(iii) But

$$(\nabla_{\underline{x}} g_j, \hat{d}) = (\nabla_{\underline{x}} g_j, -\nabla_{\underline{x}} F)$$

$$< 0 \rightarrow \hat{d} \text{ is usable}$$

by virtue of (i). This says the direction of steepest descent is a feasible direction which is also usable, contrary to the given condition.

(iv) Therefore,

$$\lambda_j > 0, \qquad j \varepsilon I_A \qquad\qquad \text{Q.E.D.}$$

This completes the proof of the Kuhn-Tucker conditions for the existence of constrained minimum of $F(\underline{x})$ subject to $g_j(\underline{x}) \leq 0$ $j = 1,\ldots,m$. They are

(i) $$g_j(\underline{x}_{OPT}) \leq 0$$

(ii) $$\lambda_j \geq 0$$

$$\lambda_j g_j(\underline{x}_{OPT}) = 0$$

(iii) $$\nabla_{\underline{x}} F(\underline{x}_{OPT}) + \sum_{j=1}^{m} \lambda_j \nabla_{\underline{x}} g_j(\underline{x}_{OPT}) = 0$$

APPENDIX C

COMPUTER SUBROUTINE LISTINGS

In order to use the IBM SHARE quasilinearization program, three user-supplied subroutines must be combined with three SHARE subroutines. The latter three are written in terms of a general problem described in section 3.5 and do not change from one problem to another. Their names and functions are:

- QUAS1 - a quasilinearization algorithm
- RECIP - a matrix inversion subroutine
- OUTPUT - a subroutine that prints out different, specified combinations of problem variables

In addition to these IBM subroutines, the user must provide three subroutines which convert a specific problem into the general formulation required by the above subroutines. Their names and functions are:

- MAIN - reads data, establishes initial guess, calls QUAS1, and terminates execution with additional output if desired
- DIFEQ - is called by QUAS1 to evaluate the differential equation and linearization gradients at every point for each iteration
- CORRC - corrects specified boundary conditions after each iteration

These six subroutines are combined into a single computer program for solving the given TPBVP.

Listings of nine subroutines are contained in this appendix. The first six represent a program for solving the minimum deflection beam problem subject to stress constraints. The order of presentation is:

MAIN

DIFEQ

CORRC

QUAS1

RECIP

OUTPUT

Following these are the three use-supplied subroutines required to solve the maximum buckling load problem subject to only geometric and material bounds. They are presented in the following order:

MAIN

DIFEQ

CORRC

```
      IMPLICIT REAL*8(A-H,O-Z)
      COMMON/QUCOMM/X,ERROR,OFFDG,H,E1MAX,E2MAX,BLO
      COMMON/DATA/BD,AC,SIG,TAU
      DIMENSION G(6),W(6),T(101),P(1),A(6,6),B(6,6),BL(6,6),
     1BR(6,6),C(6,6),CG(6,6),R(101,6),Y(101,6),D(6,6,100)
      DIMENSION PHI(8)
      IFLAG =  0
      BD    = .25
      AC    = .20
      SIG   = 8.75
      TAU   = 2.31
      N     =  6
      M     =  100
      NC    =  6
      NI    =  100
      NP    =  101
      NPAR  =  1
      NIT   = 15
      IPRNT =  1
      ERROR =  0.5D-6
      OFFDG =  5.0D-15
      BLO   =  10.
      DO 100 I=1,6
      DO 100 J=1,6
      BL(I,J) = 0.0
      BR(I,J) = 0.0
100   CONTINUE
      BL(1,1) = 1.0
      BL(2,2) = 1.0
      BL(5,5) = 1.0
      BL(6,6) = 1.0
      BR(3,3) = 1.0
      BR(4,4) = 1.0
      DO 200 I=1,NP
```

```
      T(I)    = (I-1.)/M
      Z       = T(I)
      Y(I,1) = .020*(Z**2)
      Y(I,2) = .040*Z
      Y(I,3) = .080*((1.-Z)**3)
      Y(I,4) =-.035*((1.-Z)**3)
      Y(I,5) = 13.0*(Z**4)
      Y(I,6) =-2.00*(Z**4)
200   CONTINUE
      CALL QUAS1 (N,M,NC,NI,NP,NPAR,NIT,IPRNT,G,W,T,P,A,B,BL,BR,C,DG,R,
     1Y,D,+2000)
      IF(E2MAX.LE.ERRCR) GO TO 900
800   CONTINUE
      WRITE (6,810) E2MAX,ERROR
810   FORMAT ('1'/'0',10X,'DIC NOT CONVERGE, E2MAX =',D12.5,'.GT. ERRCR
     1=',D12.5,//)
      IFLAG = -1
      GC TC 920
900   WRITE (6,910) E2MAX,ERROR
910   FORMAT ('1'/'0',10X,'CONVERGED, E2MAX = ',D12.5,'.LE. ERROR =',D12
     1.5,//)
920   CONTINUE
      WRITE(6,701)
701   FORMAT    ('0',10X,'... X AND YJ(X) AT EXIT QUAS1 ...',//)
      WRITE(6,7010)
7010  FORMAT (9X1HX,8X2HY1,8X2HY2,8X2HY3,8X2HY4,8X2HY5,8X2HY6,8X2HY7,
     18X2HY8,8X2HY9,7X3HY10//)
      WRITE(6,702) (T(I),(Y(I,J),J=1,N),I=1,NP)
702   FORMAT(5X,F7.4,6F10.4)
      IF(IFLAG.LT.0) GO TO 3000
      WRITE(6,1000)
1000  FORMAT ('1'/'0',2X,'CONTROL DEPENDENT PORTION OF HAMILTONION',//)
      WRITE (6,1100)
1100  FORMAT(T7,'X',T15,'U1',T23,'U2=.20',T37,'U2=.30',T51,'U2=.40',T65,
```

```
     1'U2=.50',T79,'U2=.60',T93,'U2=.70',T107,'U2=.80',T121,'U2=.90')
      DO 1600 I=2,11
      Z   = (I-1.)/10.
      II  = 10*(I-1) + 1
      F2X =-(1.-Z)*Y(II,3)/Y(II,6)
      DO 1600 J=1,16
      U1  = .25 + .05*(J-1.)
      DO 1400 K=1,8
      U2  = .20 + .1*(K-1.)
 1400 PHI(K)  =  U1*U2 + F2X/((U1**3)*U2)
      WRITE (6,1500) Z,U1,(PHI(L),L=1,8)
 1500 FORMAT ('0',2X,F6.3,3X,F6.3,8D14.5)
 1600 CONTINUE
      GO TO 3000
 2000 CONTINUE
      WRITE(6,2100)
 2100 FORMAT('0',10X,'ERROR EXIT 0 STRESS CONSTRAINT EXCLUDES ALL GEOMET
     1RICALLY ADMISSIBLE CONTROL')
 3000 CONTINUE
      STOP
      END
```

```
      SUBROUTINE DIFEC (N,NC,NPAR,IM,ITER,G,W,P,DG,*)
      IMPLICIT REAL*8(A-H,O-Z)
      COMMON/QUCOMM/X,ERROR,OFFDG,H,E1MAX,E2MAX,BLO
      COMMON/DATA/BD,AC,SIG,TAU
      DIMENSION G(NC),W(NC),P(NPAR),DG(NC,NC)
      DIMENSION SIGB(10),TAUB(10)
      FUNC(U1,U2,F2X) = U1*U2 + F2X/((U1**3)*U2)
      F2X = -(1.-X)*W(3)/W(6)
100   PLL = FUNC(BC,AC,F2X)
      U1  = BD
      U2  = AC
      PHI = PLL
      MPI = 0
110   US2 = (F2X/(BD**4))**.5
      IF(US2.LT.AC.OR.US2.GT.1.) GO TO 120
      PHIS= FUNC(BC,US2,F2X)
      IF(PHIS.GT.PHI)            GO TO 120
      U2  = US2
      PHI = PHIS
      MPI = 2
120   PLU = FUNC(BD,1.,F2X)
      IF(PLU.GT.PHI)             GO TO 130
      U2  = 1.
      PHI = PLU
      MPI = 0
130   CONTINUE
      US1 = (3.*F2X)**.25
      IF(US1.LT.BD.OR.US1.GT.1.) GO TO 1400
      PHIS= FUNC(US1,1.,F2X)
      IF(PHIS.GT.PHI)            GO TO 1400
      U1  = US1
      U2  = 1.
      PHI = PHIS
      MPI = 1
```

```
 1400 CONTINUE
      NB  = 0
      INB = 0
      U1BNSL = (SIG*W(3)/AC)**.5
      U1BSSL = -TAU*W(4)/AC
      U1BNSU = (SIG*W(3))**.5
      U1BSSU = -TAU*W(4)
      IF(U1BNSU.LE.1..AND.U1BSSU.LE.1.)  GO TO 1402
C
C
C     STRESS CONSTRAINTS ADMIT NO  GEOMETRICALLY ADMISSIBLE CONTROL
      NB  = -1
      WRITE(6,1401) X,U1BNSU,U1BSSU
 1401 FORMAT('0',10X,'X =',F10.4,'U1BNSU =',D15.6,'U1BSSU =',D15.6)
      GO TO 500
C
C
 1402 CONTINUE
C
C
C     STRESS CONSTRAINTS ADMIT     GEOMETRICALLY ADMISSIBLE CONTROLS
      IF(U1BNSL.GE.BD.OR.U1BSSL.GE.BD)   GO TO 1403
C     STRESS CONSTRAINTS ADMIT ALL GEOMETRICALLY ADMISSIBLE CONTROL
      NB  = 0
      GO TO 140
C
C
 1403 CONTINUE
      U2BNSU =  SIG*W(3)
      U2BSSU = -TAU*W(4)
      IF(U2BSSU.GT.U2BNSU)       GO TO 1404
      IF(U2BNSU.LT.AC)           GO TO 1410
C
C
```

```
C     U1 UPPER BOUNDARY IS 0  U2LB .LE. U2 .LE. 1.0
C     BOUNDARY IS CEFINED BY MAXIMUM NORMAL STRESS
      U2LB = U2BNSU
      INB = 1
      NB  = 2
      GO TC 140
C
C
 1404 CONTINUE
      IF(U2BSSU.LT.AC)            GO TO 1410
C
C
C     U1 UPPER BOUNDARY IS 0  U2LB .LE. U2 .LE. 1.0
C     BOUNDARY IS CEFINED BY MAXIMUM SHEAR STRESS
      U2LB = U2BSSU
      INB = 2
      NB  = 2
      GO TC 140
C
C
 1410 CONTINUE
      IF(U1BSSL.GT.U1BNSL)        GO TO 1415
C
C
C     U2 LOWER BOUNDARY IS 0  U1LB .LE. U1 .LE. 1.0
C     BOUNDARY IS CEFINED BY MAXIMUM NORMAL STRESS
      U1LB = U1BNSL
      INB = 1
      NB  = 1
      GO TC 140
C
C
 1415 CONTINUE
C
```

```
C
C      U2 LOWER BOUNDARY IS 0  U1LB .LE. U1 .LE. 1.0
C      BOUNDARY IS DEFINED BY MAXIMUM SHEAR STRESS
       U1LB = U1BSSL
       INB = 2
       NB  = 1
C
C
C
C
C      INB = 1  ...  U1(SIG,AC).GT.U1(TAU,AC)
C      INB = 2  ...  U1(SIG,AC).LT.U1(TAU,AC)
C      NB  =-1  ...  NO ADMISSIBLE CONTROL
C      NB  = 0  ...  NO ACTIVE STRESS CONSTRAINT
C      NB  = 1  ...  U1LB .LE. U1 .LE. 1   AND   U2 .EQ. AC
C      NB  = 2  ...  U2LB .LE. U2 .LE. 1   AND   U1 .EQ. 1.0
C
C
C
C
 140   PUU = FUNC(1.,1.,F2X)
       IF(PUU.GT.PHI)               GO TO 150
       U1  = 1.
       U2  = 1.
       PHI = PUU
       MPI = 0
 150   CONTINUE
       US2 = (F2X)**.5
       IF(NB.EQ.2)                  GO TO 155
       IF(US2.LT.AC.OR.US2.GT.1.) GO TO 160
 153   PHIS= FUNC(1.,US2,F2X)
       IF(PHIS.GT.PHI)              GO TO 160
       U1  = 1.
       U2  = US2
```

```
      PHI = PHIS
      MPI = 2
      IF(NB.EQ.2)                    GO TO 180
      GO TO 160
155   IF(U2LB.LT.US2.AND.US2.LT.1.) GO TO 153
      PULB= FUNC(1.,U2LB,F2X)
      IF(PULB.GT.PHI)                GO TO 180
      U1  = 1.
      U2  = U2LB
      PHI = PULB
      MPI = 3
      GO TO 180
160   PUL = FUNC(1.,AC,F2X)
      IF(PUL.GT.PHI)                 GO TO 170
      U1  = 1.
      U2  = AC
      PHI = PUL
      MPI = 0
170   CONTINUE
      US1 = (3.*F2X/(AC**2))**.25
      IF(NB.EQ.1)                    GO TO 175
      IF(US1.LT.BD.OR.US1.GT.1.) GO TO 180
173   PHIS= FUNC(US1,AC,F2X)
      IF(PHIS.GT.PHI)                GO TO 180
      U1  = US1
      U2  = AC
      PHI = PHIS
      MPI = 1
      GO TO 180
175   IF(U1LB.LT.US1.AND.US1.LT.1.) GO TO 173
      PULB= FUNC(U1LB,AC,F2X)
      IF(PULB.GT.PHI)                GO TO 180
      U1  = U1LB
      U2  = AC
```

```
      PHI = PULB
      MPI = 4
  180 CONTINUE
  200 G(1) = W(2)
      G(2) = W(3)/((U1**3)*U2)
      G(3) = W(4)
      G(4) = U1*U2
      G(5) = (1.-X)/((U1**3)*U2)
      G(6) =-W(5)
      DO 220 I=1,6
      DO 220 J=1,6
  220 DG(I,J) = 0.0
      DIU     = 0.0
      DG(1,2) = 1.0
      DG(3,4) = 1.0
      DG(6,5) =-1.0
C
C
C
C     MPI = 0  ...  U1 AND U2 ARE CONSTANTS
C     MPI = 1  ...  U1 IS NOT CONSTANT
C     MPI = 2  ...  U2 IS NOT CONSTANT
C     MPI = 3  ...  U2 IS ON A STRESS CONSTRAINT BOUNDARY
C     MPI = 4  ...  U1 IS ON A STRESS CONSTRAINT BOUNDARY
C
C
C
      MPIG= MPI + 1
      GO TO (230,240,250,260,270), MPIG
  230 DG(2,3) = 1./((U1**3)*U2)
      GO TO 290
  240 CONTINUE
      DIU     =  1./((U1**4)*U2)
      DU1W3   = .25*U1/W(3)
```

```
      DU1W6    =-.25*U1/W(6)
      DG(4,3) =  U2*DU1W3
      DG(4,6) =  U2*DU1W6
      DG(5,3) = -3.*(1.-X)*UIU*DU1W3
      DG(5,6) =  DG(5,3)*DU1W6/DU1W3
      DG(2,3) =   -3.*W(3)*UIU*DU1W3  + 1./((U1**3)*U2)
      DG(2,6) =   -3.*W(3)*UIU*DU1W6
      GO TO 290
  250 CONTINUE
      UIU      =  1./((U1**3)*(U2**2))
      DU2W3    =  .5*U2/W(3)
      DU2W6    = -.5*U2/W(6)
      DG(4,3) =  U1*DU2W3
      DG(4,6) =  U1*DU2W6
      DG(5,3) = -(1.-X)*UIU*DU2W3
      DG(5,6) =  DG(5,3)*DU2W6/DU2W3
      DG(2,3) = -W(3)*UIU*DU2W3  +  UIU*U2
      DG(2,6) = -W(3)*UIU*DU2W6
      GO TO 290
  260 CONTINUE
      IF(INB.EQ.2)  GO TO 265
      DG(2,3) = -SIG*G(2)/U2 + 1./U2
      DG(4,3) =  SIG
      DG(5,3) = -SIG*G(5)/U2
      GO TO 290
  265 CONTINUE
      DG(2,3) =  1./U2
      DG(2,4) =  TAU*G(2)/U2
      DG(4,4) = -TAU
      DG(5,4) =  TAU*G(5)/U2
      GO TO 290
  270 CONTINUE
      IF(INB.EQ.2)  GO TO 275
      DG(2,3) = -1.5*SIG*G(2)/(U1*U1*AC) + 1./((U1**3)*AC)
```

```
      DG(4,3) =  0.5*SIG/U1
      DG(5,3) = -1.5*SIG*G(5)/(U1*U1*AC)
      GO TO 290
275   CONTINUE
      DG(2,3) =  1./((U1**3)*AC)
      DG(2,4) =  3.*TAU*G(2)/(U1*AC)
      DG(4,4) = -TAU
      DG(5,4) =  3.*TAU*G(5)/(U1*AC)
290   CONTINUE
      XCS =  SIG*W(3)
      XCT = -TAU*W(4)
      U1BI=  XCS/XCT
      U2BI=  XCT**2/XCS
      DO 295 IX=1,10
      XU2 =  0.1*IX
      SIGB(IX) =  DSQRT(XCS/XU2)
      TAUB(IX) =  XCT/XU2
295   CONTINUE
      Z  =  X + .5*H
      WRITE(6,300) Z,U1,U2,F2X
300   FORMAT( 2X,'X=',D13.6,6X,'U1=',D15.6,6X,'U2=',D15.6,5X,'F2X=',D15.
     16)
      WRITE(6,310) U1BI,U2BI,INB,NB,MPI
310   FORMAT(26X,D15.6,9X,D15.6,58X,3I3)
      WRITE(6,320) (SIGB(IX),IX=1,10)
      WRITE(6,321) (TAUB(IX),IX=1,10)
320   FORMAT(2X,10D12.3)
321   FORMAT(2X,10D12.3,//)
      RETURN
500   CONTINUE
      RETURN 1
      END
```

```
SUBRCUTINE CORRC (N,M,NC,NP,NPAR,ITER,T,P,BL,BR,Y)
IMPLICIT REAL*8(A-H,O-Z)
COMMON/QUCOMM/X,ERROR,OFFDG,H,E1MAX,E2MAX,BLO
COMMON/DATA/BD,AC,SIG,TAU
DIMENSION T(NP),P(NPAR),BL(NC,NC),BR(NC,NC),Y(NP,NC)
Y(1,1)   = 0.0
Y(1,2)   = 0.0
Y(101,3) = 0.0
Y(101,4) = 0.0
Y(1,5)   = 0.0
Y(1,6)   = 0.0
RETURN
END
```

```
      SUBROUTINE QUASI(N,M,NC,NI,NP,NPAR,NIT,IPRNT,G,W,T,P,A,B,BL,BR,C,
     1DG,R,Y,D,*)
      IMPLICIT REAL*8(A-H,O-Z)
      COMMON/QUCOMM/X,ERROR,OFFDG,H,E1MAX,E2MAX,BLO
      COMMON/DATA/BC,AC,SIG,TAU
      DIMENSION G(NC),W(NC),T(NP),P(NPAR),A(NC,NC),B(NC,NC),
     1BL(NC,NC),BR(NC,NC),C(NC,NC),DG(NC,NC),R(NP,NC),
     2Y(NP,NC),D(NC,NC,NI)
      M1=M+1
      CALL OVERFL(JJJ)
      ITER=0
    1 IM = 0
      CALL CORRC (N,M,NC,NP,NPAR,ITER,T,P,BL,BR,Y)
      IF(IPRNT) 99,99,98
   98 CALL OUTPUT (N,M,NC,NP,IM,IPRNT,ITER,G,W,T,DG,Y)
   99 E2MAX = 0.0D0
      MPAS = 3
      WRITE (6,1200)
 1200 FORMAT('1'/'0',10X,'CONTROL VECTOR COMPONENTS',//)
      DO 30 IM=1,M
      IM1=IM+1
      H=T(IM1)-T(IM)
      X=T(IM)+0.5*H
      DO 2 I=1,N
    2 W(I)=0.5*(Y(IM,I)+Y(IM1,I))
      CALL DIFEQ (N,NC,NPAR,IM,ITER,G,W,P,DG,+9997)
      KKK=1
      CALL OVERFL(JJJ)
      IF(JJJ-2) 1000,1001,1001
 1000 WRITE (6,110) KKK,ITER,IM
  110 FORMAT(17H0OVERFLOW OF TYPE  I3, 12H   ITERATION I4, 10H INTERVAL
     1I4///)
 1001 DO 29 I=1,N
      W(I)=Y(IM1,I)-Y(IM,I)
```

```
      G(I)=H*G(I)
      Q= DABS(W(I)-G(I))
      IF(E2MAX-Q) 6,29,29
    6 E2MAX=Q
      I2I = IM
      J2J =  I
   29 CONTINUE
      IF(IPRNT-1)  28,28,27
   27 CALL OUTPUT (N,M,NC,NP,IM,IPRNT,ITER,G,W,T,DG,Y)
   28 H=0.5*H
      DO 5 I=1,N
      DO 5 J=1,N
    5 A(I,J)=H*DG(I,J)
      DO 44 I=1,N
      A(I,I)=1.+A(I,I)
   44 G(I)=W(I)-G(I)
      CALL RECIP (N,M,NC,IM,IPRNT,ITER,MPAS,W,A,B,C,DG)
      KKK=2
      CALL OVERFL(JJJ)
      IF(JJJ-2)  1002,1003,1003
 1002 WRITE (6,110) KKK,ITER,IM
 1003 DO 30 I=1,N
      R(IM,I)=0.
      DO 8 J=1,N
      R(IM,I)=R(IM,I)+ A(I,J)*G(J)
    8 C(I,J,IM) = -2.*A(I,J)
   30 C(I,I,IM) = 1.+ C(I,I,IM)
      IF(IPRNT) 34,34,35
   35 X = 0.5*(T(I2I) + T(I2I+1))
      WRITE (6,105) E2MAX,J2J,X
  105 FORMAT (16H0MAX. ABS. ERR.= E15.6,6H FOR Y I2,6H AT X=,F7.4)
      WRITE (6,1201) Y(NP,1)
 1201 FORMAT('0',10X,'TIP DEFLECTION  =',D20.12)
   34 IF(ITER) 31,32,31
```

```
 32 E1MAX = E2MAX
    GO TO 37
 31 IF ( E2MAX - BLC*E1MAX) 36,36,96
 96 WRITE (6,106) ITER
106 FORMAT(25HOINSTABILITY IN ITERATION I3//)
    WRITE(6,1311)
    WRITE(6,1312) (T(I),(Y(I,J),J=1,N),I=1,NP)
1311 FORMAT (4X1HX,8X2HY1,8X2HY2,8X2HY3,8X2HY4,8X2HY5,8X2HY6,8X2HY7,
   18X2HY8,8X2HY9,7X3HY10//)
1312 FORMAT (1X,F7.4,6F10.4)
 97 RETURN
 36 IF (E2MAX - ERRCR) 97,97,37
 37 IF (ITER - NIT) 137,97,97
137 IF (E2MAX - .01*E1MAX) 38,38,33
 38 IF (E2MAX - PRERR) 33,33,97
 33 ITER=ITER+1
    PRERR = E2MAX
    MPAS = 5
    DO 111 I=1,N
    G(I)=0.
    DO 10  J=1,N
 10 A(I,J)=0.
111 A(I,I)=1.
    DO 12 IM=1,M
    IF (MOD(IM,2)) 212,312,212
212 X = -1.
    GO TO 152
312 X= 1.
152 DO 52 I=1,N
    DO 52 J=1,N
 52 G(I)=G(I)-X*A(I,J)*R(IM,J)
    DO 56 I=1,N
    DO 56 J=1,N
    B(I,J)=0.
```

```
      DO 56 K=1,N
   56 B(I,J)= B(I,J) + A(I,K)*C(K,J,IM)
      DO 12 I=1,N
      DO 12 J=1,N
   12 A(I,J)=B(I,J)
      IF (MOD(M,2)) 412,512,412
  412 X = -1.
      GO TO 157
  512 X = 1.
  157 DO 57 I=1,N
      R(M1,I) = 0.
      DO 57 J=1,N
      R(M1,I)=R(M1,I)-BL(I,J)*G(J)
   57 B(I,J)=X*A(I,J)
      DO 67 I=1,N
      DO 67 J=1,N
      A(I,J)=0.
      DO 68 K=1,N
   68 A(I,J)=A(I,J)+BL(I,K)*B(K,J)
   67 A(I,J)=A(I,J)+BR(I,J)
      KKK=3
      CALL OVERFL(JJJ)
      IF(JJJ-2) 1004,1005,1005
 1004 WRITE (6,110) KKK,ITER,M1
 1005 CALL RECIP (N,M,NC,M1,IPRNT,ITER,MPAS,W,A,B,C,DG)
      KKK=4
      CALL OVERFL(JJJ)
      IF(JJJ-2) 1006,1007,1007
 1006 WRITE (6,110) KKK,ITER,M1
 1007 DO 14 I=1,N
      G(I)=0.
      DO 14 J=1,N
   14 G(I)=G(I)+A(I,J)*R(M1,J)
      DO 15 I=1,N
```

```
      Y(M1,I) = Y(M1,I) + G(I)
   15 R(M1,I) = G(I)
      CO 40 IM=1,M
      ID=M1-IM
      ID1=IC+1
      CO 16 I=1,N
      G(I)=0.
      CC 16 J=1,N
   16 G(I) = G(I) - D(I,J,ID)*R(ID1,J)
      CO 40 I=1,N
      R(IC,I)=R(ID,I)+G(I)
   40 Y(IC,I)=Y(ID,I)+R(IC,I)
      KKK=5
      CALL OVERFL(JJJ)
      IF(JJJ-2) 1008,1,1
 1008 WRITE (6,110) KKK,ITER,M1
      GO TC 1
 9997 RETURN 1
      END
```

```
      SUBROUTINE RECIP (N,M,NC,IM,IPRNT,ITER,MPAS,W,A,B,C,DG)
      IMPLICIT REAL*8(A-H,O-Z)
      COMMON/QUCOMM/X,ERROR,OFFDG,H,E1MAX,E2MAX,BLO
      COMMON/DATA/BD,AC,SIG,TAU
      DIMENSION W(NC),A(NC,NC),B(NC,NC),C(NC,NC),DG(NC,NC)
      DO 1 I=1,N
      DO 1 J=1,N
    1 C(I,J)=A(I,J)
      DO 70 I=1,N
      FACT=A(I,I)
      A(I,I)=1.
      IF(FACT) 72,34,72
   34 WRITE (6,100) I,IM
  100 FORMAT(22HOZERO ON DIAGONAL ROW I3,  7HMATRIX  I3///)
      STOP
   72 DO 71 J=1,N
   71 A(I,J)=A(I,J)/FACT
      DO 70 K=1,N
      IF(K-I) 73,70,73
   73 IF(A(K,I)) 74,70,74
   74 FACT=A(K,I)
      A(K,I)=0.
      DO 75 J=1,N
   75 A(K,J)=A(K,J)-FACT*A(I,J)
   70 CONTINUE
      NPAS=1
   14 SUM=0.
      DO 2 I=1,N
      DO 2 J=1,N
      DG(I,J)=0.0
      DO 2 K=1,N
    2 DG(I,J)=DG(I,J)+A(I,K)*C(K,J)
      N1=N-1
      DO 4 I=1,N1
```

```
      I1=I+1
      CO 4 J=I1,N
    4 SUM= SUM+ DABS(CG(I,J))+ DABS(DG(J,I))
      IF( IPRNT-3)  31,32,32
   32 WRITE (6,101) SUM
  101 FORMAT(30H SUM CF ABS OF  OFFDIAG TERMS  E15.8)
   31 IF (SUM - OFFDG) 13,13,12
   12 IF (NPAS -1) 9,9,15
   15 PRERR = 2.*PRERR
      IF (SUM - PRERR) 8,200,2C0
  200 DO 201 I=1,N
      CO 201 J=1,N
  201 A(I,J) =  B(I,J)
      IF (IPRNT - 3) 13,203,203
  203 WRITE (6,102)
  102 FORMAT(30H USED NEXT TO LAST RECIPROCAL.)
      GO TC 13
    8 IF (NPAS-MPAS) 9,13,13
    9 PRERR = SUM
      NPAS  = NPAS  +1
      CO 202 I = 1,N
      CD 202 J = 1,N
  202  B(I,J) = A(I,J)
      CO 7 I=1,N
    7 CG(I,I)=CG(I,I)-1.
      CO 10 I=1,N
      CO 11 J=1,N
      W(J)=0.
      CO 11 K=1,N
   11 W(J)=W(J)+A(K,I)*DG(J,K)
      CO 10 J=1,N
   10 A(J,I)=A(J,I)-W(J)
      GC TC 14
   13 RETURN
```

```
      SUBROUTINE OUTPUT (N,M,NC,NP,IM,IPRNT,ITER,G,W,T,DG,Y)
      IMPLICIT REAL*8(A-H,O-Z)
      COMMON/QUCOMM/X,ERROR,OFFDG,H,E1MAX,E2MAX,BLO
      COMMON/DATA/BC,AC,SIG,TAU
      DIMENSION G(NC),W(NC),T(NP),DG(NC,NC),Y(NP,NC)
      IF (IM) 1,1,16
    1 WRITE (6,100) ITER
      WRITE (6,102)
      M1 = M+1
      DO 6 I=1,M1
    6 WRITE (6,104) T(I),(Y(I,J),J=1,N)
      IF (IPRNT-2) 21,11,11
   11 WRITE (6,106) ITER
      WRITE (6,108)
      WRITE (6,110)
      GO TO 21
   16 WRITE (6,112) X,(G(J),J=1,N)
      WRITE (6,112) X,(W(J),J=1,N)
      IF (IPRNT-3)  21,21,18
   18 DO 19 I=1,N
   19 WRITE (6,114) (DG(I,J),J=1,N)
   21 RETURN
  100 FORMAT (25H1SOLUTION AFTER ITERATION I3//)
  102 FORMAT (4X1HX,8X2HY1,8X2HY2,8X2HY3,8X2HY4,8X2HY5,8X2HY6,8X2HY7,
     18X2HY8,8X2HY9,7X3HY10//)
  104 FORMAT (1XF7.4,10F10.4)
  106 FORMAT (19H1ERRORS IN ITERATE I4//)
  108 FORMAT (4X1HX,6X4HH*G1,6X4HH*G2,6X4HH*G3,6X4HH*G4,6X4HH*G5,
     16X4HH*G6,6X4HH*G7,6X4HH*G8,6X4HH*G9,5X5HH*G10)
  110 FORMAT (4X1HX,7X3HDY1,7X3HDY2,7X3HDY3,7X3HDY4,7X3HDY5,7X3HDY6,
     17X3HDY7,7X3HDY8,7X3HDY9,6X4HDY10///)
  112 FORMAT (1XF7.4,10F10.4)
  114 FORMAT (1X10F10.3)
      END
```

```
      IMPLICIT REAL*8(A-H,O-Z)
      REAL*8 MU,KM,ML,LAM
      COMMON/QUCOMM/X,ERROR,OFFDG,H,E1MAX,E2MAX,BLO
      COMMON/DATA/ AU,AL,EU,EL,RU,RL,MU,KM,DU,SU,LAM
      COMMON/TCO/ U1,US1,ASF,XHT
      DIMENSION G(7),W(7),T(201),P(10),A(7,7),B(7,7),
     1          BL(7,7),BR(7,7),C(7,7),DG(7,7),R(201,7),
     2          Y(201,7),D(7,7,200)
      DIMENSION DYSAVE(201,7)
C     ...  MUST SPECIFY  ML = FCT(X)
      ML  =  0.0
      U1  =  1.0
      AU  =  1.1
      EU  =  1.0
      RU  =  1.0
      AL  =  0.9
      EL  =  1.0
      RL  =  1.0
      MU  =  1.0
      DU  =  2.5
      SU  =  0.5
      KM  =  .012
      N      = 7
      NC     = 7
      M      = 200
      NI     = 200
      NP     = 201
      NPAR   =  1
      NIT    =  5
      IPRNT  =  1
      ERROR  = 1.0D-8
      OFFDG  = 5.0D-15
      BLO    = 10.
      DO 100 I=1,7
```

```
      DO 100 J=1,7
      BL(I,J) = 0.0
      BR(I,J) = 0.0
  100 CONTINUE
      BL(1,1) = 1.0
      BL(2,2) = 1.0
      BR(3,3) = 1.0
      BR(4,4) = 1.0
      BR(5,5) = 1.0
      BL(6,6) = 1.0
      BL(7,7) = 1.0
  150 CONTINUE
      PI  = 3.141596
      PI2 = PI**2
      PO2 = .5*PI
      LAM =  2.74950
      DO 200 I=1,NP
      T(I)    =  (I-1.)/M
      Z       = T(I)
      CT      =  DCOS(PO2*Z)
      ST      =  DSIN(PO2*Z)
      Y(I,1) =  1.0 - CT
      Y(I,2) =  1.75*ST
      Y(I,3) =  2.50*CT
      Y(I,4) = -LAM  -  ( 1. -Z )*KM/MU
      Y(I,5) = -.500*CT
      Y(I,6) =  .350*ST
      Y(I,7) = -.275*ST
  200 CONTINUE
      CALL QUAS1 (N,M,NC,NI,NP,NPAR,NIT,IPRNT,G,W,T,P,A,B,BL,BR,C,DG,R,
     1Y,C,DYSAVE)
      IF(E2MAX.LE.ERROR) GO TO 900
  800 CONTINUE
      WRITE (6,810) E2MAX,ERROR
```

```
810  FORMAT ('1'/'0',10X,'DID NOT CONVERGE, E2MAX =',D12.5,'.GT. ERROR
    1=',D12.5)
     GO TO 920
900  WRITE (6,910) E2MAX,ERROR
910  FORMAT ('1'/'0',10X,'CONVERGED, E2MAX = ',D12.5,'.LE. ERROR =',D12
    1.5)
920  CONTINUE
     CIF = -( LAM + Y(1,4) )*MU/KM
     XHT =  1./M
     YI2 =  0.0
     DO 930 I=1,NP
     DYT =  Y(I,2)*Y(I,2)*XHT
     IF(NP-I) 925,925,930
925  DYT =  .5*DYT
930  YI2 =  YI2 + DYT
     ASF =  DSQRT(YI2)
     AIN =  1./ASF
     DO 935 J=1,N
     DO 935 I=1,NP
     GO TO (931,931,931,935,933,933,935), J
931  Y(I,J) =  AIN*Y(I,J)
     GO TO 935
933  Y(I,J) =  ASF*Y(I,J)
935  CONTINUE
     WRITE (6,945) CIF
945  FORMAT ('1'/'0',20X,'STATE VARIABLES ... COST FUNCTION =',D15.6)
     WRITE (6,946) YI2,ASF
946  FORMAT('0',20X,'NORM SQUARED ...',D15.6,5X,'NORMALIZATION CONSTANT
    1 ...',D15.6,//)
     WRITE (6,947)
947  FORMAT (9X1HX,8X2HY1,8X2HY2,8X2HY3,8X2HY4,8X2HY5,8X2HY6,8X2HY7,
    18X2HY8,8X2HY9,7X3HY10//)
     WRITE (6,950) (T(I),(Y(I,J),J=1,7),I=1,NP)
950  FORMAT(5X,F7.4,7F10.4)
```

```
      STOP
      END
```

```
      SUBROUTINE DIFEQ (N,NC,NPAR,IM,ITER,G,W,P,DG)
      IMPLICIT REAL*8(A-H,O-Z)
      REAL*8 MU,KM,ML,LAM
      COMMON/QUCOMM/X,ERROR,OFFDG,H,E1MAX,E2MAX,BLO
      COMMON/DATA/ AU,AL,EU,EL,RU,RL,MU,KM,DU,SU,LAM
      COMMON/TCO/ U1,US1,ASF,XHT
      DIMENSION G(NC),W(NC),P(NPAR),DG(NC,NC)
C     ...  MUST SPECIFY  ML = FCT(X)
      ML  =  0.0
      EXU =  1./3.
      US2 =  1.0
      US3 =  1.0
      XN  =  2.*W(5)*W(3)
      XD  =  (KM*W(7) - MU)*US2*US3
      ARG =  XN/XD
      Z1  =  DSIGN(US2,ARG)
      US1 =  ( DABS(ARG) )**EXU
      US1 =  Z1*US1
      IF(US1.GT.AL) GO TO 60
 50   U1   = AL
      DUW3 = 0.0
      DUW5 = 0.0
      DUW7 =  0.0
      GAM1 = 0.0
      GO TO 100
 60   IF(US1.GE.AU) GO TO 70
      U1   =  US1
      DUW3 =  US1/(3.*W(3))
      DUW5 =  US1/(3.*W(5))
      DUW7 = -EXU*US1*KM/(KM*W(7) - MU)
      GAM1 =  0.0
      GO TO 100
 70   CONTINUE
      U1   =  AU
```

```
      DUW3 = 0.0
      DUW5 = 0.0
      DUW7 = 0.0
      GAM1 =  0.0
100   CONTINUE
      U2   =  US2
      U3   =  US3
      GAM2 =  0.0
      GAM3 =  0.0
      G(1) =  W(2)
      G(2) =  W(3)/(U1*U1*U2)
      G(3) =  W(2)*W(4)
      G(4) =  KM*(ML + U1*U3)
      G(5) = -W(6)*W(4)
      G(6) = -W(5)/(U1*U1*U2)
      G(7) = -W(6)*W(2)
200   CONTINUE
      DO 220 I=1,7
      DO 220 J=1,7
220   DG(I,J) =  0.0
      DG(1,2) =  1.0
      DGU     = -2.*G(2)/U1
      DG(2,3) =  DGU*DUW3   +  1./(U1*U1*U2)
      DG(2,5) =  DGU*DUW5
      DG(2,7) =  DGU*DUW7
      DG(3,2) =  W(4)
      DG(3,4) =  W(2)
      DGU     =  KM*U3
      DG(4,3) =  DGU*DUW3
      DG(4,5) =  DGU*DUW5
      DG(4,7) =  DGU*DUW7
      DG(5,4) = -W(6)
      DG(5,6) = -W(4)
      DGU     = -2.*G(6)/U1
```

```
      DG(6,3) =  DGU*DUW3
      DG(6,5) =  DGU*DUW5   -  1./(U1*U1*U2)
      DG(6,7) =  DGU*DUW7
      DG(7,2) = -W(6)
      DG(7,6) = -W(2)
  300 CONTINUE
      RETURN
      END
```

```
SUBROUTINE CORRC (N,M,NC,NP,NPAR,ITER,T,P,BL,BR,Y)
IMPLICIT REAL*8(A-H,O-Z)
REAL*8 MU,KM,ML,LAM
COMMON/QUCOMM/X,ERROR,OFFDG,H,E1MAX,E2MAX,BLO
COMMON/DATA/ AU,AL,EU,EL,RU,RL,MU,KM,DU,SU,LAM
DIMENSION T(NP),P(NPAR),BL(NC,NC),BR(NC,NC),Y(NP,NC)
Y(1,1)  = 0.0
Y(1,2)  = 0.0
Y(NP,3) = 0.0
Y(NP,4) =-LAM
Y(NP,5) = 0.0
Y(1,6)  = 0.0
Y(1,7)  = 0.0
RETURN
END
```

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BIOGRAPHICAL SKETCH

Jerry C. Hornbuckle was born in Clearwater, Florida, in 1942. On graduating from Stranahan High School in Fort Lauderdale, he attended the Aerospace Engineering program of the University of Florida. Studies there were interrupted by a year's employment at Honeywell's inertial guidance division in St. Petersburg, Florida. He received his BSAE from the University of Florida in 1965.

Following graduation he was employed by the Boeing Company in Huntsville, Alabama. While there he worked on error analyses, the Apollo-Saturn V on-board computer flight program, and trajectory simulation. During the evenings he was a part-time student at the University of Alabama in Huntsville, receiving an MSE in Engineering Mechanics in 1971.

In 1969 he began doctoral studies in the Engineering Sciences department of the University of Florida. Although specializing in dynamics and vibrations, his dissertation research was in the field of structural optimization. He expects to graduate in August, 1974, with a Ph.D. in Engineering Mechanics.

Mr. Hornbuckle is married to the former Carolyn A. Smith of Fort Lauderdale and has two children. He enjoys golf, tennis, racquetball, music, and reading. In the area of reading, he is interested in psychology and history, especially military and aviation history.

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

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William H. Boykin, Jr., Chairman
Associate Professor of Engineering Sciences

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

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